

Structural Modeling of Rigid Pavement with Load-rate Dependent Subgrade

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Abstract

It has been observed by many researchers that the subgrade behavior is load rate-dependent. The apparent subgrade stiffness is much higher under the fast moving axle loading than during the slow developing temperature curling and moisture warping. However, traditionally a single k-value is used for both curling and axle load analysis. This paper presents a mathematically rigorous solution for determining stresses and deflections in rigid pavements using separate k-values for slab curling and axle loading. The developed procedure illustrates that, in the event that the k-value for the temperature curling analysis is less than that of the axle loading, the overall stress state is lower under combined axle and thermal loads. At the same time, the paper demonstrates that due to the inherent non-linear behavior of the interaction of the slab and foundation, the use of simple linear superposition of curling and axle loading stresses for two different k-values may result in significant errors.

Introduction

The mechanistic-based design procedures for rigid pavements commonly utilize Westergaard's model (Westergaard 1926, Westergaard 1927) for determination of structural responses such as stresses and deflections. This model treats a pavement as a medium-thick Kirchhoff plate resting on a Winkler (spring) foundation. Westergaard derived solutions for the various loading conditions such as a wheel load applied at the slab interior, edge, or corner. He also obtained solutions for curling of infinite and semi-infinite slabs subjected to a linear temperature gradient throughout the slab's thickness. To develop these solutions Westergaard introduced significant simplifications, such as infinite slab size, single slab (no effect of the adjacent slabs), single placed layer (no base or subbase), single wheel load (no multiple wheel axles), and uniform support (no separation of the concrete slab from the subgrade) (Ioannides et al. 1999).

The introduction of finite element programs for rigid pavement analysis such as KENSLAB (Huang 1993) and ILLI-SLAB (Tabatabae 1977) permitted addressing important limitations of the Westergaard model. A series of improvements in ILLI-SLAB over almost two decades (Ioannides 1984, Korovesis 1990, Khazanovich 1994) permitted the analysis of multi-slab, multi-layered rigid pavements subjected to a combined action of a nonlinear gradient and multi-wheel loading. The latest version of ILLI-SLAB, ILSL2, is also capable to account for the effect of separation between the pavement layers as well as the pavement layers and subgrade. A commercial version of ILLI-SLAB, ISLAB2000, had been extensively used during the development of the Mechanistic-Empirical Pavement Design Guide (Khazanovich et al. 2000).

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One of the limitations of Westergaard's solutions, however, has not received sufficient attention in the past. It has been observed by many researchers that the behavior of subgrade is dependent on the rate of loading (Chatti et al 1994). Hall et al. (1997) demonstrated that the subgrade stiffness backcalculated from falling-weight deflectometer testing is approximately twice as high as the subgrade stiffness derived from the slowly applied plate load test. This implies that subgrade stiffness for the case of fast moving axle loads is higher than subgrade stiffness under slowly developing thermal loads (such as temperature curling and moisture warping). If a rigid pavement is subjected to a combination of axle loading and temperature curling, the conventional use of a single coefficient of subgrade reaction (k -value) for the characterization of subgrade behavior is inadequate. This phenomenon has also been recognized in the recently revised Illinois Department of Transportation (IDOT) design procedure (Bordelon et al. 2009) which allows the user to use two different k -values: one for analysis of axle loading and another for analysis of temperature curling. Nevertheless, there is no mathematically rigorous yet practical analytical method permitting analysis of rigid pavements using two different k -values.

The main challenge in developing a solution using different k -values for axle and temperature loadings is the effect of separation of the concrete slab from the foundation. This makes the slab-foundation interaction non-linear – in other words, the stresses and deflections obtained for the axle loading cannot be simply added to the stresses and deflections obtained for the temperature curling (Tabatabae 1977, Ioannides 1984, Khazanovich et al. 2000). Ioannides (2005) demonstrated that a simple addition of the solutions for axle loading and slab curling may result in a significant error, especially for a night-time conditions, even if a single k -value is used for both axle loading and curling analyses. The IDOT design procedure (Bordelon et al. 2009) does not address this issue by using a superposition correction factor developed for a single k -value.

This paper presents a mathematically rigorous solution for determining stresses and deflections in rigid pavements using different k -values for slab curling and axle loading. The solution is implemented in a finite element code which can be considered as a generalization of the widely used ISLAB2000.

Characterization of Subgrade Behavior

Finite Element Formulation. The FE model developed under this research provides a generalization of the current framework for the commercial package ISLAB2000 (Khazanovich et al. 2000). The Kirchhoff-Love plate theory is used to model the bending of the constructed layers as an isotropic and homogenous medium-thick plate (Timoshenko and Woinowsky-Krieger 1959).

ISLAB2000 incorporates many subgrade models (such as Winkler [1867], Pasternak [1954], and Kerr-Vlasov [1964]) that provide the foundation support for the slab. Among them, the Winkler foundation is the simplest foundation model, which is defined using a proportionality constant between the applied pressure and plate deflection at any point. The spring formulation of the Winkler foundation is adopted to model the foundation support. This formulation of the Winkler foundation models the foundation with concentrated springs at the nodes of the plate element as shown in Figure 1 (Khazanovich et al. 2000).

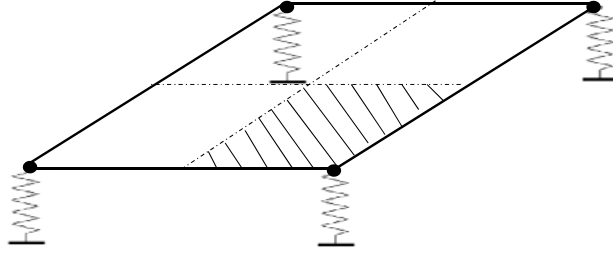


Figure 1. Spring idealization of Winkler foundation using concentrated spring at the nodes of the plate element.

In the finite element program developed, a four-node rectangular finite element is used, where each node has three degrees of freedom. The third degree of freedom for each node corresponds to the vertical deflection. The stiffness matrix for the foundation has non-zero elements only along the diagonal in locations corresponding to the vertical displacements of each node. These non-zero values are indicated in Equation 1 below.

$$[K_{df}^{foundation}]_e = \begin{cases} k_{subgrade} * \frac{A_e}{4} * C(w_d) & \text{for } d = f \text{ and } d \in \{3, 6, 9, 12\} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where

subscript e denotes an individual element in the plate,

$[K^{foundation}]$ is the stiffness matrix of the foundation supporting the plate element,

$k_{subgrade}$ is the coefficient of subgrade reaction,

A_e is the area of the element,

d and f are local degrees of freedom at the element,

w_d is the vertical displacement for degree of freedom d , and

$C(w_d)$ is a function indicating the slab-foundation contact condition

$$C(w_d) = \begin{cases} 1 & \text{if } w_d \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

This approach is similar to that of ISLAB2000 to model the separation of the slab from the foundation for curling analysis. This approach can be summarized by noting that for an out-of-contact node, no contribution from the foundation is considered (Khazanovich et al. 2000). This may make the boundary value problem nonlinear. When a pavement is subjected to compressive loads, the deformation of the foundation increases linearly, with an increase in the surface pressure. However, the foundation cannot resist vertical upward movement and will separate from the slab (causing the presence of a void) under tensile loads. As a result, during the day time, curling of the slab because of positive temperature gradient may cause a void under the center of slab. Similarly, during the night time, negative temperature gradient may cause a void under the edges of the slab. This implies that due to non-linear interaction of slab with the foundation, the two different loading cases (and resulting stresses) cannot be linearly superimposed to mimic the combined loading.

The foundation stiffness matrix $[K_{foundation}]_e$ is added to the element stiffness matrix $[k]_e$ in order to incorporate the boundary conditions for the slab-on-grade analysis when the element node is in contact with the foundation.

$$[K] = \sum_{e=1}^{ne} ([k]_e + [K_{foundation}]_e) \quad (3)$$

where

$[K]$ is the global stiffness matrix, and
 ne is the total number of elements over the plate mesh.

Similar to the global stiffness matrix, the global force vectors are also assembled by adding the terms of the element force vectors at corresponding global degrees of freedom over all the elements. Finally, the nodal displacements of the plate are calculated by solving the equilibrium equation (Zienkiewicz and Taylor 1967).

Subgrade Characterization Using 2-k Values.

As mentioned above, the use of a single coefficient of subgrade reaction (k-value) for characterizing the subgrade behavior under a combination of loads with different loading durations seems to be an oversimplification. Consider a rigid pavement subjected to an arbitrary temperature distribution throughout the thickness of the concrete slab acting on time interval $0 < t < t_T$. The pavement is also subjected to an axle loading that acts instantaneously at the end of the same time interval. It is proposed that two separate coefficients of subgrade reactions (k) should be considered as follows:

1. Static coefficient of subgrade reactions, k_{static} – to characterize the subgrade response when the pavement is subjected to temperature loads, and
2. Dynamic coefficient of subgrade reactions, $k_{dynamic}$ – to characterize the subgrade response when the pavement is subjected to axle loads.

As discussed above, the stresses and deflections obtained using these separated k-values cannot be superimposed due to the nonlinear behavior of the slab-foundation interaction. To address this challenge, the following rigorous solution is proposed. First, assuming a static subgrade stiffness of k_{static} , curling analysis will be conducted to determine stresses and deflections in the slab prior to axle loading. Next, an inverse problem is solved for the slab on a subgrade having stiffness $k_{dynamic}$ and the same deflection basin as that determined for the thermal problem in the previous step. This involves determining a fictitious load and the corresponding stress state in the slab. Finally, the system is solved for the combined action of the axle load and the fictitious load determined from the previous step. This determines the change in stress in the slab due to axle loading.

In summary, an analysis of a slab-on-grade with two k-values (k_{static} and $k_{dynamic}$) requires the solution of three boundary value problems (BVPs). The first BVP considers curling of the slab under temperature loading only and uses the static modulus of subgrade reaction, k_{static} to characterize the subgrade behavior. The second elastic BVP involves determination of the stress field in the pavement such that the pavement has the same deflection profile as was determined in the first elastic solution but the subgrade behavior is now characterized by the dynamic modulus of

subgrade reaction, $k_{dynamic}$. In the third elastic BVP, the dynamic modulus of subgrade reaction, $k_{dynamic}$ is used to determine the deflections and stresses from the combined effect of temperature curling and axle loading. The total stresses in the pavement are computed as a combination of the stresses from these three solutions, using the medium-thick plate theory (Zienkiewicz and Taylor 1967).

The First Boundary Value Problem. Consider problem 1 – a concrete slab rests on the spring idealization of a Winkler foundation. The modulus of subgrade reaction is equal to the static modulus, k_{static} . The pavement system of problem 1 is subjected to a positive temperature gradient $T(z)$ only as shown in Figure 2.

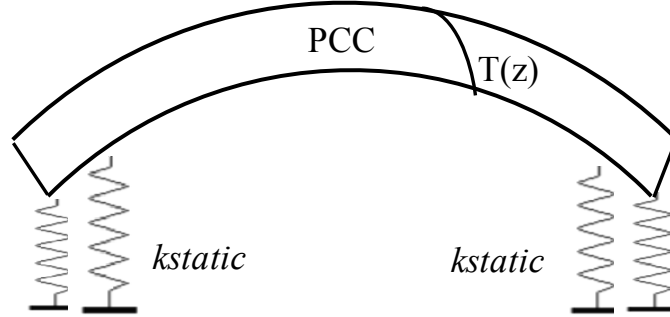


Figure 2. System 1 under positive temperature gradient $T(z)$ only.

The equilibrium equation for system 1 can be expressed as follows:

$$[K_1]\{\delta_1\} = \{F_{therm}\} \quad (4)$$

where

$[K_1]$ is the global stiffness matrix for system 1 given by equations (1) and (3) when $k_{subgrade}$ is equal to the static modulus k_{static} ,
 $\{F_{therm}\}$ is the global force vector due to temperature distribution $T(z)$, and
 $\{\delta\}$ is the global displacement vector of system 1.

The displacements of the slab, δ_1 can be written as:

$$\{\delta_1\} = [K_1]^{-1} \{F_{therm}\} \quad (5)$$

Since system 1 is subjected to temperature distribution $T(z)$, elastic stress in an element of system 1 can be calculated as:

$$\{\sigma_{1BVP}\}_e = [\overline{D}] (\{\epsilon_1\} - \{\epsilon_{therm}\})_e \quad (6)$$

where

$\{\sigma_{1BVP}\}$ is the elastic stress from the first elastic solution,
 $[\overline{D}]$ is the material property matrix,
 $\{\epsilon_1\}$ is the total strain corresponding to the global displacements δ_1 , and
 $\{\epsilon_{therm}\}$ is the thermal strain due to temperature distribution $T(z)$.

The Second Boundary Value Problem. Consider problem 2 that has the same structure and material properties as problem 1, except that the subgrade is characterized using the dynamic modulus of subgrade reaction, $k_{dynamic}$. Assume that a fictitious force F_{fict} acts on the pavement system such that its deflection profile is exactly the same as that from problem 1, i.e. the deflection at each node of system 2 is exactly equal to the deflection at the corresponding node in system 1 (as indicated in Figure 3).

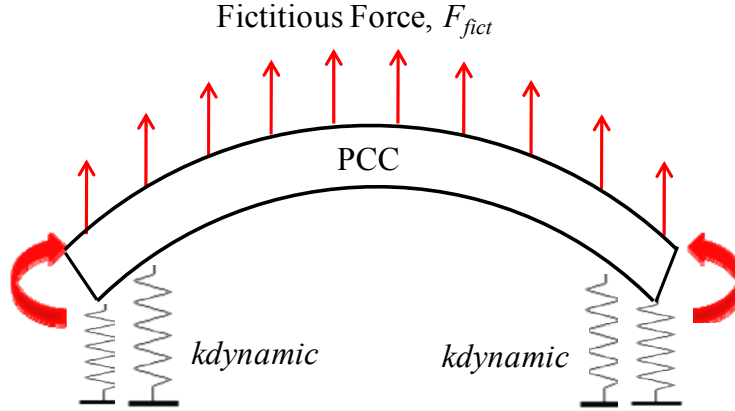


Figure 3. System 2 under fictitious force F_{fict} .

Since the global displacements of problem 2 are assumed to be exactly the same as those from problem 1, the fictitious force F_{fict} can be computed as follows:

$$\{F_{fict}\} = [K_2] \{\delta_1\} \quad (7)$$

where

$[K_2]$ is the global stiffness matrix for system 2 given by equations (1) and (3) when $k_{subgrade}$ is equal to the dynamic modulus $k_{dynamic}$.

Since no initial strains act on system 2 and the global displacements of system 2 are exactly the same as those of system 1, the elastic stress in an element of system 2, $\{\sigma_{2BVP}\}$ can be calculated as follows:

$$\{\sigma_{2BVP}\}_e = [\bar{D}] \{\varepsilon_1\}_e \quad (8)$$

The Third Boundary Value Problem. The third BVP has the same structure as system 2, but here the load vector is equal to the sum of the fictitious load vector F_{fict} obtained in step two and the load vector associated with the axle load F (illustrated in Figure 4).

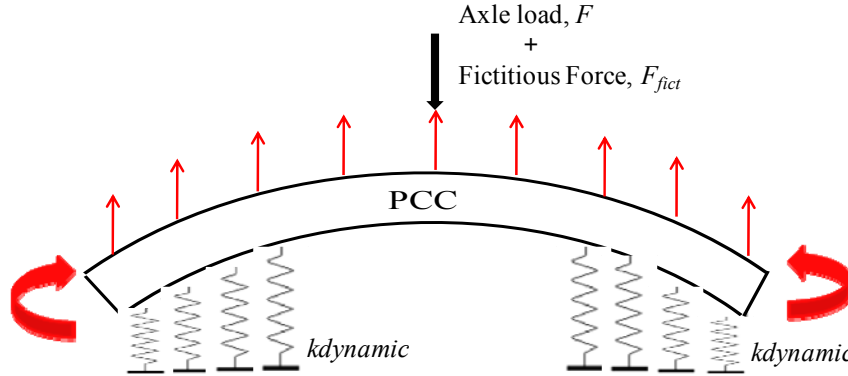


Figure 4. System 2 under mid-slab traffic load F and fictitious force F_{fict} .

The stress at the bottom of the PCC layer at the mid-slab location due to the total load is denoted as σ_{3BVP} . The global displacements δ under a combination of loads can be computed as follows:

$$\{\delta\} = [K_2]^{-1} (\{F\} + \{F_{fict}\}) \quad (9)$$

where

$\{F\}$ is the global force vector due to traffic loads, and
 $\{F_{fict}\}$ is the global fictitious force vector from the second elastic solution.

The elastic stress from the third elastic problem can be calculated as follows:

$$\{\sigma_{3BVP}\}_e = [\bar{D}] \{\epsilon_T\}_e \quad (10)$$

where

$\{\sigma_{3BVP}\}$ is the elastic stress from the third elastic solution, and
 $\{\epsilon_T\}$ is the total strain corresponding to the global displacements δ .

Combined Stress. Finally, to obtain the stress distribution in the pavement due to the combined effects of temperature and axle loading, the change in stresses in system 2 from the axle load should be added to the curling stress. Therefore, the solutions of the three problems are combined as follows:

$$\sigma_{2k} = \sigma_{1BVP} + (\sigma_{3BVP} - \sigma_{2BVP}) \quad (11)$$

where

σ_{2k} is the combined stress at a given location,
 σ_{1BVP} is the stress at the given location from the first elastic solution,
 σ_{2BVP} is the stress at the given location from the second elastic solution, and
 σ_{3BVP} is the stress at the given location from the third elastic solution.

Two examples presented below shall illustrate this procedure as well as demonstrate its validity and importance.

Illustrative Examples

Two simple examples are provided to illustrate the importance of the concepts detailed above. The first example examines the sensitivity of the pavement responses to the static k -value. This analysis demonstrates that if the static k -value is equivalent to the dynamic k -value, the method above reproduces the ISLAB2000 solution. However, ignoring the fact that the static k -value is lower than the dynamic k -value leads to an overestimation of the stress state in rigid pavements experiencing both thermal and axle loading. The second example will illustrate that a simple superposition of stresses resulting from thermal and axle loading cannot provide an accurate representation of the stress state.

Both examples below consider a 15 ft long, 12 ft wide, and 8 in thick concrete slab placed on a Winkler foundation. The modulus of elasticity, Poisson's ratio, unit weight, and coefficient of thermal expansion of concrete are taken as equal to $4.00\text{E}+06$ psi, 0.15, 0.087 lb/in^3 , and $5.50\text{E}-06\text{ }1/^{\circ}\text{F}$, respectively. To model the night-time loading conditions, the pavement is subjected to a -30° (negative) linear temperature gradient throughout the slab's thickness and is loaded with single-axle dual-wheel (SADW) loads placed at both the edges of the slab as shown in figure 5(a). To model the day-time loading conditions, the pavement is subjected to a 30° (positive) linear temperature gradient throughout the slab's thickness and loaded with single-axle dual-wheel (SADW) load at the center of the slab as shown in figure 5(b). Each wheel has a tire footprint of 7 in x 7 in and the axle has a total load of 18 kips. A uniform mesh consisting of 6 in x 6 in elements is generated over the plate. The dynamic and static moduli of subgrade reaction for the Winkler foundation are taken as equal to 200 psi/in and 100 psi/in, respectively. The stresses and deflections are evaluated at point A, as highlighted in Figure 5.

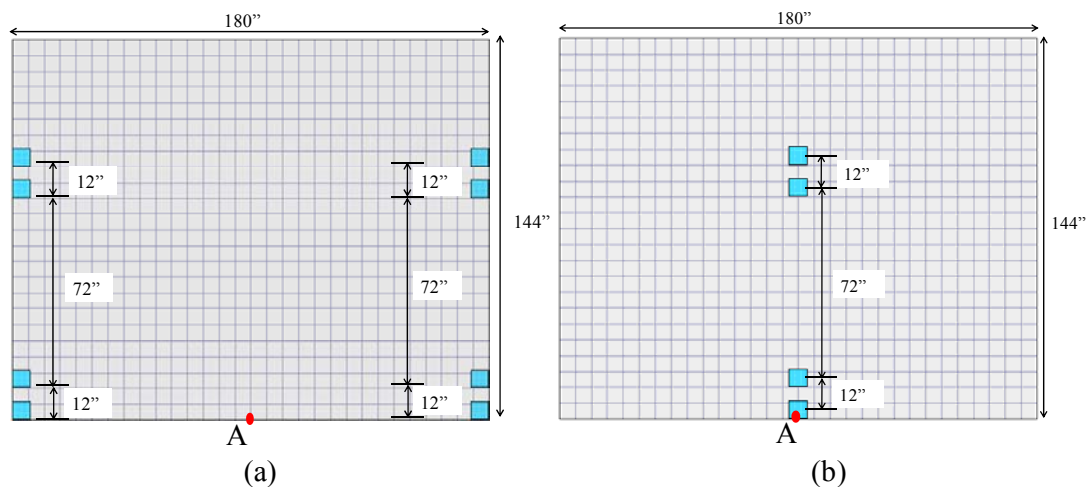


Figure 5. Mesh and load configuration for the composite pavement subjected to (a) night-time loading conditions and (b) day-time loading conditions.

Example 1. Consider the rigid pavement on Winkler foundation presented above. The slab geometry, material properties, loading conditions, and mesh are exactly the same as presented for combined axle loading and temperature gradient (night-time loading). The dynamic subgrade modulus $k_{dynamic}$ is set equal to 200 psi/in. The

static subgrade modulus k_{static} is varied between 100 psi/in and 200 psi/in, in increments of 20 psi/in. Figure 6 presents the stress in the top edge of the concrete layer at the mid-slab location with respect to the static subgrade modulus k_{static} .

Figure 6 also plots the stress obtained from the same rigid pavement subjected to the night-time combined axle loading and temperature gradient, analyzed using ISLAB2000. The modulus of subgrade reaction (single value input) is set equal to 200 psi/in.

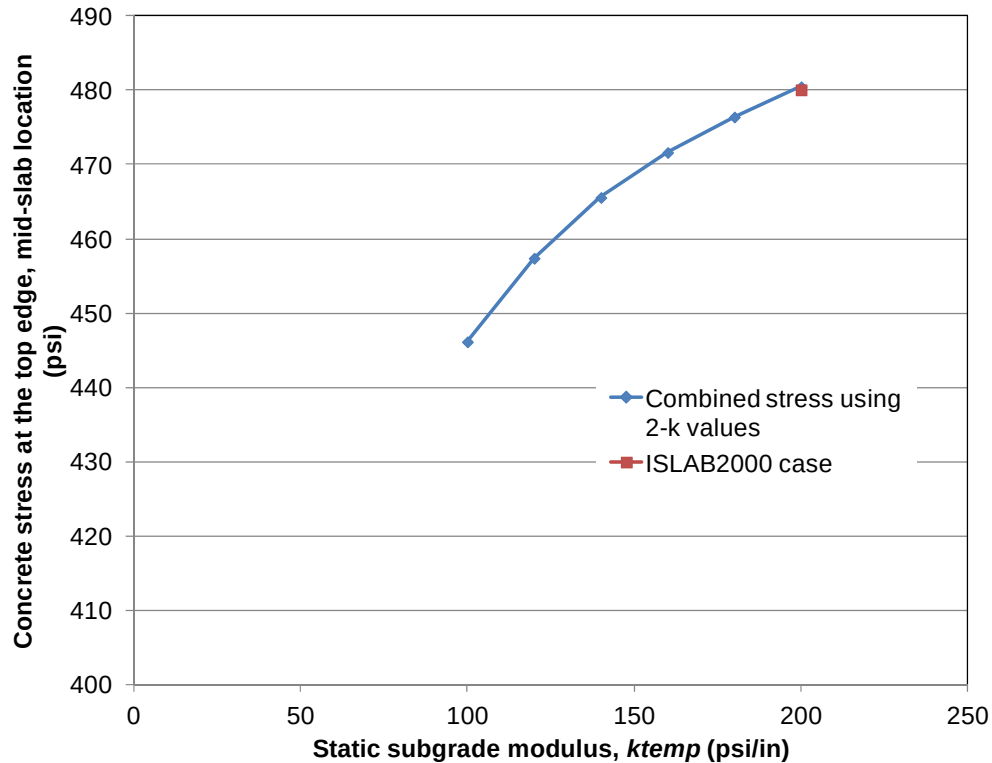


Figure 6. Stress at the top of the concrete layer at the slab edge with respect to the static subgrade modulus, for night-time loading conditions.

It is observed from Figure 6 that the stress in the concrete layer under the combined loading increases monotonically with an increase in the static subgrade modulus. Also, for the case when the static subgrade modulus k_{static} is equal to the dynamic subgrade modulus $k_{dynamic}$, the stress computed using the combined stress approach matches the stress computed from the ISLAB2000 case. This confirms the robustness of the developed finite element procedure. It can also be observed that the use of the static k-value equal to half of the dynamic k-value would result in a reduction of predicted stresses by 40 psi for the given case. Therefore, for this example the current stress computation procedure implemented in the Mechanistic-Empirical Pavement Design Guide is overly conservative.

Example # 2. Table 1 presents the deflections, stress, and combined concrete stress at the bottom of the concrete slab at mid-slab location, using the combined stress procedure (equation [11]) involving the 2- k values, for the day-time loading conditions. The table also presents the deflection, stress, and a simple addition of stresses when the pavement is subjected to the same (a) temperature gradient only with the static subgrade modulus k_{static} and (b) axle loads only with the dynamic subgrade modulus $k_{dynamic}$. Similarly, Table 2 presents the deflections and stresses

for the night-time loading conditions at the top of the concrete slab at mid-slab location. A difference between the stresses computed using the combined stress procedure and the simple addition (i.e. [a] + [b]) is also noted for both the loading conditions.

Table 1. Deflections and stresses at the bottom of the concrete layer at the mid-slab edge location (A) for day-time loading conditions.

	Deflection, in	Longitudinal Stress, psi
2-k values		
BVP # 1, <i>kstatic</i> , temperature loads only	0.01095	170.78
BVP # 2, <i>kdynamic</i> , fictitious loads only	0.01095	-217.45
BVP # 3, <i>kdynamic</i> , axle loads + fictitious loads	0.03577	176.56
Combined stress (equation [11])		564.79
Simple Addition		
(a) <i>kstatic</i> , temperature load only	0.01095	170.78
(b) <i>kdynamic</i> , traffic load only	0.01664	319.72
Sum: (a) + (b)		490.50
% Difference between 2-k values and Simple Addition		15.15%

Table 2. Deflections and stresses at the top of the concrete layer at the mid-slab edge location (A) for night-time loading conditions.

	Deflection, in	Longitudinal Stress, psi
2-k values		
BVP # 1, <i>kstatic</i> , temperature loads only	-0.01620	112.82
BVP # 2, <i>kdynamic</i> , fictitious loads only	-0.01620	-275.23
BVP # 3, <i>kdynamic</i> , axle loads + fictitious loads	-0.00775	57.89
Combined stress (equation [11])		446.94
Simple Addition		
(a) <i>kstatic</i> , temperature load only	-0.01620	112.82
(b) <i>kdynamic</i> , traffic load only	0.00011	166.49
Sum: (a) + (b)		279.31
% Difference between 2-k values and Simple Addition		59.66%

The difference between the stresses from the combined stress computation procedure using 2-k values and the simple addition highlights the importance of the developed procedure. This is particularly clear in the case of night-time loading conditions, as is the case for the single k-value analysis (Ioannides 2005). This example has obvious implications for other attempts to use two k-values for different loading conditions, such as the IDOT correction factor. However, the superposition correction factor is applicable only for day-time loading, as detailed in Bordelon et al. (2009).

Conclusions

Although it has been widely recognized that the foundation stiffness is much higher under dynamic loads (as opposed to static loads), the majority of design procedures use a single k-value for the prediction of stresses and deflections in concrete pavements. For example, the Mechanistic-Empirical Pavement Design Guide utilizes a dynamic k-value for stress computation. Recently, IDOT has adopted a design procedure that utilizes different k-values for temperature curling and axle load analysis. While this adoption is a step in the right direction, the procedure used by the IDOT design oversimplifies the stress computation.

This paper presents a mathematically rigorous yet practical and accessible procedure to estimate the stresses using different k-values for loading corresponding to temperature curling and axle loads. If these k-values are equivalent, then the procedure results in stress predictions identical to those of ISLAB2000. However, in the event that the k-value for the temperature curling analysis is less than that of the axle loading, the overall stress state is lower under combined axle and thermal loads. At the same time, the paper demonstrated that due to the inherent non-linear behavior of the interaction of the slab and foundation, the use of simple linear superposition of curling and axle loading stresses for two different k-values may result in significant errors.

References

- Bordelon, A., Roesler, J., and Hiller, J. (2009). "Mechanistic-empirical design concepts for jointed plain concrete pavements in Illinois." *Illinois Department of Transportation Report No. FHWA-ICT-09-052*, Springfield, IL.
- Chatti, K., Lysmer, J., and C.L. Monismith (1994). "Dynamic Finite-Element Analysis of Jointed Concrete Pavements." *Transportation Research Record: Journal of the Transportation Research Board No. 1449*, Washington, DC: Transportation Research Board, 79–90.
- Hall, K.T., Darter, M.I., Hoerner, T.E., and Khazanovich, L. (1997). "LTPP Data analysis phase I: validation of guidelines for k-value selection and concrete pavement performance prediction." *Office of Engineering and Highway Operations R&D Report No. FHWA-RD-96-198*, Federal Highway Administration, McLean, VA.
- Huang, Y. H. (1993). *Pavement analysis and design*, Englewood Cliffs, NJ: Prentice Hall.
- Ioannides, A.M. (1984). *Analysis of slabs-on-grade for a variety of loading and support conditions*, Ph.D. Thesis, Department of Civil Engineering, University of Illinois, Urbana-Champaign.
- Ioannides, A.M., Davis, C.M. and Weber, C.M. (1999). "Westergaard curling solution reconsidered." *Transportation Research Record: Journal of the Transportation Research Board No. 1684*, Washington, DC: Transportation Research Board, 61–70.
- Ioannides, A.M. (2005). "Stress prediction for cracking of jointed plain concrete pavements, 1925–2000 An overview." *Transportation Research Record: Journal of the Transportation Research Board No. 1919*, Washington, DC: Transportation Research Board, 47–53.

- Kerr A. D. (1964). "Elastic and viscoelastic foundation models." *ASME Journal of Applied Mechanics*, 31(3), pp. 491-498.
- Khazanovich, L. (1994). *Structural analysis of multi-layered concrete pavement systems*. Ph.D. Thesis, University of Illinois, Urbana-Champaign.
- Khazanovich, L., Yu, H.T., Rao, S., Galasova, K., Shats, E. and Jones, R. (2000). "ISLAB2000—finite element analysis program for rigid and composite pavements: User's guide." *ARA Inc., ERES Consultants Division*, Champaign IL.
- Korovesis, G.T. (1990). *Analysis of slab-on-grade pavement systems subjected to wheel and temperature loadings*. Ph.D. Thesis, Department of Civil Engineering, University of Illinois, Urbana-Champaign.
- Pasternak P. L. (1954). "Fundamentals of a new method of analysis of structures on elastic foundation by means of two subgrade coefficients." Gosudarstvennoe Izdatel'stvo Literatury po Stroitel'stvu i Arkhitekture, Moscow (in Russian).
- Tabatabaie-Raissi, A.M. (1977). *Structural analysis of concrete pavement joints*. Ph. D. Thesis, Department of Civil Engineering, University of Illinois, Urbana-Champaign.
- Timoshenko S. P. and Woinowsky-Krieger S. (1959). *Theory of plates and shells*. 2nd edition, McGraw-Hill, NY.
- Westergaard, H. M. (1926). "Stresses in concrete pavements computed by theoretical analysis." *Public Roads*, 7 (2), 25–35.
- Westergaard, H. M. (1927). "Analysis of stresses in concrete pavements due to variations of temperature." *Proceedings: Highway Research Board*, National Research Council, Washington, DC: Transportation Research Board, Vol. 6, 201–217.
- Winkler, E. (1967). "Die lehre von der elastizitat und festigkeit (Theory of elasticity and strength)." H. Dominicus, Prague (in German).
- Zienkiewicz O. C. and Taylor R. L. (1967). *Finite element method for solid and structural mechanics*, 1st Edition, Elsevier.