

THICKNESS DESIGN OF CONCRETE PAVEMENTS BY PROBABILISTIC METHOD

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A probabilistic finite element computer program was developed for the practical design of concrete pavements. Based on the variability of traffic, material and geometric parameters, the cracking index, which is currently being used as a design criterion in most of the mechanistic methods, is converted into reliability, so the effect of variability on thickness design can be evaluated. It is shown that the use of reliability as a design criterion results in a more reasonable and realistic design. The program can be applied to slabs on liquid, solid or layered foundations. Results obtained from different types of foundations are compared and a manual method for estimating reliability is presented.

The fatigue principle has been used throughout the world for the design of concrete pavements. Given the axle loads of various groups and the properties of concrete and foundation, the most critical tensile stresses in the concrete slab under various axle loads can be determined from the theory of elasticity. The allowable number of stress repetitions under each axle load group is related to the stress ratio, which is a quotient of the stress and the modulus of rupture of the concrete. Based on the Miner's hypothesis of cumulative damage in fatigue (1), the cracking index, c , of the pavement at the end of design life can be expressed as

$$c = \sum_{i=1}^m \frac{n_i}{N_i} \quad (1)$$

in which m = number of axle load groups, n_i = predicted number of stress repetitions for load i during the design life, and N_i = allowable number of stress repetitions for load i . The design is considered satisfactory if the cracking index is less than or equal to 1. The above method has long been used by the Portland Cement Association (PCA), although an erosion criterion based on corner deflections has been added in the 1984 edition of the PCA design manual (2).

The use of the above deterministic method results in a cracking index so sensitive to the thickness of the slab to make it incomprehensible. For example, a cracking index of 50% is obtained for a slab thickness of 8 in. (203 mm). If the thickness is increased to 8.5 in. (216 mm), the cracking index is decreased to less than 10%. If the thickness is decreased to 7.5 in. (190 mm), the cracking index is increased to more than 100%. It does not appear reasonable that a slight change in thickness should affect the success or failure of the design to such an extent. In fact, the cracking index depends on a large number of traffic, material and geometric parameters. It is difficult to select these parameter values because they exhibit a wide range of variations. If average values are used, the reliability of the design is only 50 percent and is usually not considered sufficient. To be on the safe side, designers tend to use more conservative values. If conservative values are used for every parameter, the final design will become too conservative. The arbitrary selection of these parameter values may result in widely different designs. These difficulties can be overcome if a probabilistic approach is employed.

The purpose of this paper is to present a probabilistic finite element method for the practical design of concrete pavements based on the fatigue principle. Although the probabilistic concept has been applied to the AASHTO empirical equation in the 1986 design guide (3) and in the VESYS computer program for the analysis of flexible pavements (4), the procedure has not been incorporated, to the authors' knowledge, in any of the mechanistic computerized methods for the design of concrete pavements. It will be shown that the use of probabilistic procedure results in a more realistic and reasonable design.

ASSUMPTIONS

Before the presentation of the method, it is necessary to discuss some assumptions connected with the mechanistic methods of

thickness design, so that the limitations of the method are fully understood. First of all, it should be pointed out that the design of concrete pavements is a very complex process and involves so many factors which are difficult to evaluate. Therefore, some simplifying assumptions must be made in order to obtain a solution.

Warping Stresses

It is generally agreed that the maximum stress due to wheel loads occurs at the edge near the midslab (2,5). Therefore, the stress at the slab edge due to temperature warping is of great interest. Based on the field measurements at the Arlington and AASHO Road Tests (6,7), it is reasonable to assume that the maximum temperature gradient in concrete pavement is 3.50°F per in. (0.077°C per mm) of slab thickness during the day when the slab is warped down and about half the above amount at night when it is warped up. Downward warping is more critical not only because the stresses are larger due to greater temperature gradients but also because they are additive to the loading stresses, whereas the warping and loading stresses are compensative for upward warping.

Figure 1 shows the effect of slab length on warping stresses for an 8-in. (203 mm) slab with a Young's modulus of 4×10^6 psi (27.6 Gpa), a Poisson's ratio of 0.15, and a modulus of subgrade reaction of 100 pci (27 MN/m³). A temperature differential of 28°F (15.6°C) was assumed for downward warping and 14°F (7.8°C) for upward warping. Both cases of full and partial contact were considered. The results for partial contact were obtained by the KENSLABS computer program (8) using a slab width of 16 ft (4.9 m). The results for full contact were obtained from Bradbury's chart (9) based on Westergaard's analysis. The use of partial

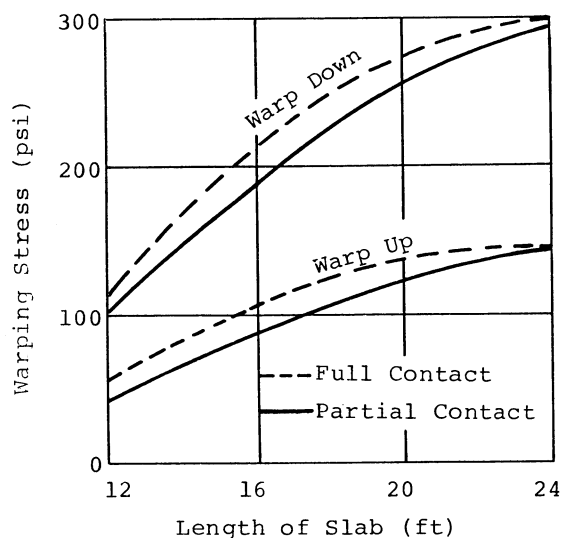


Figure 1. Stresses due to temperature warping. (1 ft=0.3 m, 1 psi=6.9 kPa)

contact is more realistic because when the slab warps down, the interior of the slab is not in contact with the subgrade; whereas when the slab warps up, the edge is not in contact.

As can be seen from Figure 1, warping stresses at the slab edge based on partial contact are smaller than those based on full contact, but the difference is not considered very significant. For a 24 ft (7.3 m) long slab, downward warping results in a stress of 294 psi (2.03 Mpa), which is only slightly smaller than the 340 psi (2.35 Mpa) caused by the 18 kip (80 kN) single axle load. However, similar to the PCA design method (2), warping stresses are not considered in this paper due to the following reasons:

1. Joints and steel are used to relieve and take care of warping stresses. Warping stresses are relieved when the concrete cracks. Minute cracks will not affect the load-carrying capacity of pavements as long as the load transfer across cracks can be maintained.

2. When the fatigue principle is used for design, it is not practical to combine loading and warping stresses. A pavement may be subjected to millions of load repetitions during the design period, but the number of stress reversals due to warping is quite limited.

3. Warping stresses may be added to or subtracted from loading stresses to obtain the combined stresses. If the design is governed by the edge stress, warping stresses should be added to loading stresses during the day but subtracted from the loading stresses at night. Due to this compensative effect and the fact that a large number of heavy trucks is driven at night, it may not be that critical if warping stresses are ignored.

4. Previous experience has demonstrated that thickness design based on load alone yields satisfactory results.

Loading Stresses

If the warping stress can be ignored, the only stress to be considered will be the loading stress at the slab edge. It is well known that warping and pumping cause the loss of subgrade contact and should have some effects on loading stresses. Fortunately, the effect of contact conditions on the edge stress is not very significant. Therefore, similar to the PCA design method (2), the loss of subgrade contact was not considered in this paper. However, a stress adjustment factor was used in the computer program to reflect the effect of warping, pumping and the load placement relative to the slab edge.

Figure 2 shows the effect of slab length and warping on loading stresses caused by an 18 kip (80 kN) single axle load. The slabs are the same as those used in Figure 1 and subject to the same temperature differentials but only the

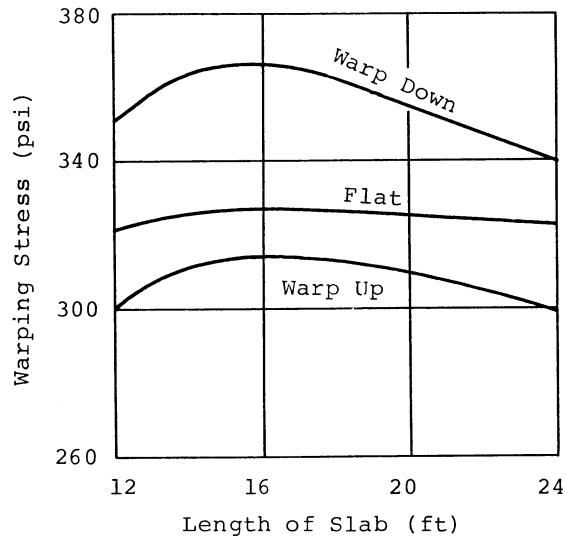


Figure 2. Stresses due to loading.
(1 ft=0.3 m, 1 psi=6.9 kPa)

stresses due to loading are included. It can be seen that the loss of subgrade contact causes an increase in the edge stress when the slab warps down but a decrease in the edge stress when the slab warps up. This is reasonable because downward warping causes the loss of subgrade contact in the interior of the slab and thus increases the subgrade reaction at the edge, while upward warping reduces the subgrade reaction at the edge. This phenomenon is completely different from the case of corner loading where upward warping causes a drastic increase in both stresses and deflections. As the length of slab has only a small effect on the edge stress, a 20 ft (6.1 m) slab was employed for all the analysis presented in this paper.

Deflections

It is generally recognized that slab deflection and drainage are important factors in rigid pavement design because they will cause the pumping of subgrade soils or the blowing of subbase materials. However, in the probabilistic and mechanistic method of thickness design presented herein, both deflection and drainage are not considered not because they are unimportant but because the state of the art has not advanced to the point where they can be treated theoretically. The difficulty in analyzing pavement deflection is that the most critical deflection occurs at the slab corner and is affected to a large extent by the slab-subgrade contact, which varies with temperature differentials, length of slab, joint design, type of subbase, and subsurface drainage. This difficulty can be greatly reduced when the edge stress, instead of the corner deflection, is considered. For this

reason, only the method based on fatigue cracking caused by the edge stress is presented in this paper. Designers in using this method should make sure that proper drainage, subbase and load transfer across the joints are provided, so that the pavement will not fail by pumping or erosion.

DESCRIPTION OF METHOD

Finite Element Method

The probabilistic finite element computer program named PMRPD (Probabilistic Method of Rigid Pavement Design) described in this paper is a modification of the KENSLABS program presented at the previous conference (8). KENSLABS can be applied to multiple slabs of nonuniform thickness with load transfer across the joints and the effect of temperature warping with subsequent loss of subgrade contact can also be analyzed. To facilitate applications and reduce the program size, PMRPD considers only one slab of uniform thickness and eliminate the tedious process of checking subgrade contact because, as discussed previously, the contact conditions have only a small effect on the edge stress. The reason that only one slab is needed is because the most critical stress condition occurs when the load is applied near the pavement edge far away from the joint, so the efficiency of load transfer across the joint has practically no effect on the maximum stress obtained.

In the analysis, the slab is considered as an elastic plate on a liquid, solid or layer foundation. After dividing the slab into rectangular finite elements, the relationship between the displacements and the forces at the nodes can be written as

$$R[K]\{\delta\} = q\{F\} \quad (2)$$

in which R = modulus of rigidity of the plate; $[K]$ = stiffness matrix of the slab-foundation system when $R=1$; $\{\delta\}$ = nodal displacements, including a vertical deflection and two rotations; q = contact pressure, which is equal to the total load divided by the contact area; and $\{F\}$ = nodal forces when $q=1$. Note that

$$R = \frac{E_c h^3}{12(1-\nu_c^2)} \quad (3)$$

in which E_c = Young's modulus of plate, h = thickness of plate and ν_c = Poisson's ratio of plate; and

$$[K] = [K_p] + \frac{k}{R} [K_f] \quad (4)$$

in which $[K_p]$ = stiffness matrix of plate when $R=1$; k = modulus of subgrade reaction of liquid foundation, elastic modulus of solid foundation, or elastic modulus of the first layer in layer foundation; and $[K_f]$ = stiffness matrix of foundation when $k=1$. The reason that R , k and q

are factored out from $[K_p]$, $[K_c]$ and $\{F\}$, respectively, is to solve the following simultaneous equation with only one random variable, k/R .

$$[K]\{\delta'\} = \{F\} \quad (5)$$

in which $\{\delta'\}$ = displacement factors which must be multiplied by q/R to obtain the nodal displacements. The probabilistic solution will become more difficult if more than one random variable exist in Eq. 5. After the displacements, $\{\delta'\}$, are determined, the stresses, $\{s\}$, can be obtained by

$$\{s\} = \frac{6q}{h^2} [S]\{\delta'\} \quad (6)$$

in which $[S]$ = stress matrix when $R=1$. To obtain the stresses, $[S]$ must be multiplied by R and $\{\delta'\}$ divided by R . Because R cancels out, it does not appear in Eq. 6. In Eq. 6, $h^2/6$ is the section modulus of the plate.

Probabilistic Procedure

In the probabilistic approach, every random variable has a mean and a variance. The random variables include slab thickness, h , contact pressure, q , Young's modulus of concrete, E_c , subgrade modulus, k , modulus of rupture of concrete, M , and predicted number of load repetitions, n . By the use of Taylor's series, it can be shown that the variance of a function $g(x_1, x_2, \dots, x_m)$ consisting of m independent variables x_i can be determined by (10)

$$V[g] = \sum_{i=1}^m \left(\frac{\partial g}{\partial x_i} \right)^2 V[x_i] \quad (7)$$

in which $V[g]$ = variance of g , $V[x_i]$ = variance of x_i , μ = mean value of each x to be substituted into the function in parenthesis. In Eq. 5, the only random variable is the relative stiffness, r , defined as k/R in Eq. 4, or

$$r = \frac{12(1 - \nu_c^2) k}{E_c h^3} \quad (8)$$

Based on Eq. 7 and the independence of h , E_c and k

$$V[r] = \{12(1 - \nu_c^2)\}^2 \left\{ \left(\frac{3k}{E_c h^4} \right)^2 V[h] + \left(\frac{k}{2h^3} \right)^2 V[E_c] + \left(\frac{1}{E_c h^3} \right)^2 V[k] \right\} \quad (9)$$

If the variables are dependent, the variance of g becomes

$$V[g] = \sum_{i=1}^m \sum_{j=1}^m \left(\frac{\partial g}{\partial x_i} \right) \left(\frac{\partial g}{\partial x_j} \right) \text{cov}(x_i, x_j) \quad (10)$$

in which $\text{cov}(x_i, x_j)$ is the covariance of x_i and x_j . When $i=j$, $\text{cov}(x_i, x_j) = V[x_i] = V[x_j]$; when $i \neq j$, $\text{cov}(x_i, x_j) = \sqrt{V[x_i] V[x_j]}$. If two

Variables x_1 and x_2 are functions of r , Eq. 10 can be written as

$$V[g] = \left\{ \left(\frac{\partial g}{\partial x_1} \right) \sqrt{V[x_1]} + \left(\frac{\partial g}{\partial x_2} \right) \sqrt{V[x_2]} \right\}^2 \quad (11)$$

in which $V[x_1]$ and $V[x_2]$ can be determined from $V[r]$ using Eq. 7.

In solving Eq. 5, every term involving r will have a corresponding variance. If $a = b/c$, from Eq. 11

$$V[a] = V\left[\frac{b}{c}\right] = \left(\frac{1}{c}\right)^2 V[b] - \frac{b}{c^2} \sqrt{V[c]}^2 \quad (12)$$

If $a = b - cd$, from Eq. 11

$$V[a] = V[b - cd] = \left(\sqrt{V[b]} - d\sqrt{V[c]} - c\sqrt{V[d]}\right)^2 \quad (13)$$

Although $V[a]$ is always positive, the term within the parenthesis on the right side of Eqs. 12 and 13 may be positive or negative. Because this term is used when a is combined with other terms, Eqs. 12 and 13 should be replaced by

$$D[a] = D\left[\frac{b}{c}\right] = \frac{1}{c} D[b] - \frac{b}{c^2} D[c] \quad (14a)$$

$$D[a] = D[b - cd] = D[b] - d D[c] - c D[d] \quad (14b)$$

in which D is the first derivative or the variance before square. Note that

$$V[a] = (D[a])^2 \quad (15)$$

and $D[a]$ may be positive or negative. The application of the above concept for solving simultaneous equations by the Gauss elimination method is presented in the Appendix. In Eq. 6, let

$$\{s'\} = [S]\{\delta'\} \quad (16)$$

After δ' and $D[\delta']$ are obtained by solving Eq. 5, s' and $D[s']$ at each node can be determined by

$$s' = \sum_{i=1}^m s_i \delta'_i \quad (17)$$

$$D[s'] = \sum_{i=1}^m s_i D[\delta'_i] \quad (18)$$

in which m = number of surrounding nodes and s_i = element of stress matrix $[S]$. From Eqs. 6 and 16

$$s = \frac{6qs'}{h^2} \quad (19)$$

$$V[s] = 36 \left\{ \left(\frac{s'}{h^2} \right)^2 V[q] + \left(\frac{q}{h^2} \right)^2 (D[s'])^2 + \left(\frac{2qs'}{h^3} \right)^2 V[h] - 2 \left(\frac{2qs'}{h^3} \right) \left(\frac{q}{h^2} \right) (fD[s']\sqrt{V[h]}) \right\} \quad (20)$$

in which f = fraction of $D[s']$ contributed by the variation of h and can be obtained from Eq. 9, or

$$f = \frac{36(1-\nu_c^2)k}{E_c h^4} \sqrt{\frac{V[h]}{V[r]}} \quad (21)$$

The first three terms on the right side of Eq. 20 indicates that q , s' and h are independent and the last term is the correlation between h and s' because both contain the variation of h .

After the maximum stress and its variance are found, the allowable number of repetitions, N , can be determined from the stress ratio, s/M , where s is the critical tensile stress and M is the modulus of rupture. The following equations suggested by PCA (11) can be used.

$$\text{When } \frac{s}{M} \geq 0.55 \quad \log N = \frac{0.9718 - \frac{s}{M}}{0.0828} \quad (22a)$$

$$\text{When } 0.45 < \frac{s}{M} < 0.55 \quad N = \left(\frac{4.2577}{\frac{s}{M} - 0.4325} \right)^{3.268} \quad (22b)$$

$$\text{When } \frac{s}{M} \leq 0.45 \quad N = \text{unlimited} \quad (22c)$$

The variance of s/M

$$V\left[\frac{s}{M}\right] = \left(\frac{1}{M}\right)^2 V[s] + \left(\frac{s}{M^2}\right)^2 V[M] \quad (23)$$

The variance of $\log N$ based on Eq. 22a

$$V[\log N] = 145.86 V\left[\frac{s}{M}\right] \quad (24a)$$

or

$$V[N] = \frac{N^2}{0.1886} V[\log N] \quad (24b)$$

Although $V[N]$ for $s/M < 0.55$ can be easily determined from Eq. 22b, this variance becomes so large and yields unreasonable results, especially when s/M is small. For this reason, Eq. 24 is used also for $s/M < 0.55$. It should be noted that the fatigue equations are not deterministic but also exhibit a range of variations, so a variance due to the inadequacy of fatigue equations can be added to Eq. 24b. However, due to the lack of information, this additional variance term was not used in this paper.

Note that Eq. 22c is valid only in a deterministic sense. In the probabilistic method, the stress ratio itself has a variance and the probability that the ratio be greater

than 0.45 always exists even though the mean stress ratio is much smaller than 0.45. From Eq. 22b, the allowable number of repetitions for a mean stress ratio of 0.45 is 6.28×10^7 , which corresponds to a probability of 50%. If the mean stress ratio is smaller than 0.45, the probability that it exceeds 0.45 can be determined by assuming the stress ratio as a normal distribution. Dividing 3.14×10^7 by this probability, which is always smaller than 0.5, will give the allowable number of repetitions for stress ratios smaller than 0.45.

Given the predicted number of load repetitions, n , and its variance, the cracking index can be obtained from Eq. 1 and the variance of cracking index by

$$V[c] = \sum_{i=1}^m \left\{ \left(\frac{1}{N_i} \right)^2 V[n_i] + \left(\frac{n_i}{N_i^2} \right)^2 V[N_i] \right\} \quad (25)$$

Because load repetitions are always expressed in log scale, it is reasonable to assume that cracking index is also a log normal distribution. The variance of $\log c$ can be determined from the variance of c by

$$V[\log c] = \frac{0.1886}{c^2} V[c] \quad (26)$$

The standard deviation of $\log c$ is

$$\sigma_{\log c} = \frac{0.4343}{c} \sqrt{V[c]} \quad (27)$$

Knowing the mean of $\log c$ and the standard deviation of $\log c$, the reliability of the design is indicated by the shaded area in Figure 3 and defined as

$$\text{Reliability} = 1 - \text{Probability}(\log c \geq 0) \quad (28)$$

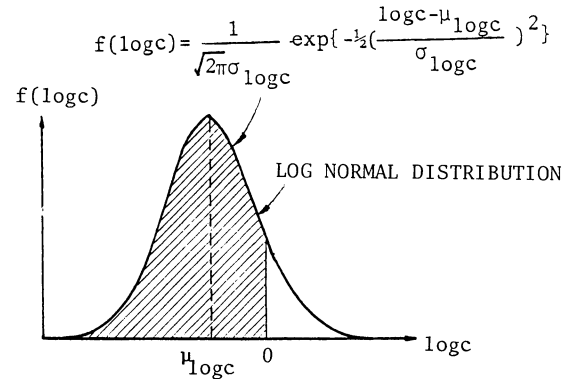


Figure 3. Reliability of cracking index.

Computer Program

The computer program was initially written in FORTRAN 77 for an IBM 3084 mainframe computer but was later converted to an IBM PC. The program has the following features which make it particularly useful as a practical design tool:

1. The program can be applied to a liquid, solid or layer foundation. With the ever-increasing speed and storage of personal computers, it is no longer necessary to assume the foundation as a liquid with a fictitious k value. The more realistic solid or layer foundation consisting of a maximum of 6 layers can be used.

2. The program can be applied to mixed traffic consisting of single, tandem and tridem axle loads. Starting from the heaviest axle load in one group, the analysis will stop or switch to the next group if the stress ratio falls below 0.25. It was found that when the stress ratio equals 0.25, the probability that it exceeds 0.45 is negligibly small. The program was so written that the stiffness matrix is reduced to an upper triangular matrix only once. After the matrix is reduced, very little time is required to analyze additional loads.

3. The program can determine the reliability of different trial thicknesses in the same run. For solid and layer foundations, this capability can also save some computer time because the stiffness matrix of the foundation obtained for the first thickness can be used directly for the remaining thicknesses.

4. The program can generate the finite element grid automatically based on the axle load data, as shown in Figure 4. Because of symmetry, only one half of the slab is considered. In all the examples to be presented, the following dimensions were used: distance between dual tire, $D_a = 13.5$ in. (0.34 m), distance between wheels, $D_w = 75$ in. (1.91 m), distance between tandem axles, $D = 48$ in. (1.22 m), width of slab, $W = 190$ in. (4.83 m), and half length of slab, $L = 120$ in. (3.05 m). The width of tire, T_w , was determined from the first or the heaviest axle load. If desired, the users can also specify their own grid instead of using the standard grid.

PRESENTATION OF RESULTS

Comparison with PCA Manual

The 1984 edition of the PCA design manual (2) contains an example for fatigue analysis of a concrete pavement. It is interesting to compare the cracking indices presented in the manual with those obtained by PMRPD. The results are shown in Table 1. The original data contains a total of 20 axle loads, 10 for single and 10 for tandem. As the last three tandem

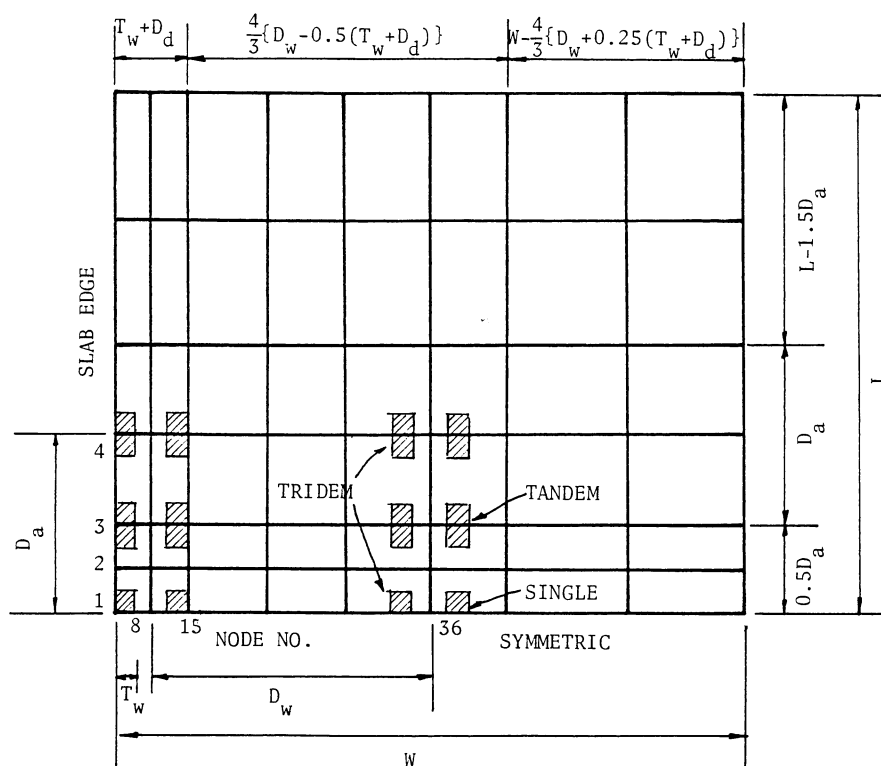


Figure 4. Division of slab into finite elements.

Table 1. Comparison of fatigue cracking with PCA manual

Thickness 9.5 in. Modulus of Rupture 650 psi			Subgrade Modulus k 130 pci Load Safety Factor 1.2		
Axle Load (kip)	Multiplied by 1.2	Expected Repetitions	By PMRPd		By PCA
			Stress Ratio	Cracking Index (%)	Cracking Index (%)
Single Axle					
30	36.0	6,310	0.597	18.88	23.3
28	33.6	14,690	0.565	17.85	19.1
26	31.2	30,140	0.532	14.02	13.1
24	28.8	64,410	0.498	7.67	5.4
22	26.4	106,900	0.463	1.08	0
20	24.0	235,800	0.428	0.29	0
18	21.6	307,200	0.391	0.21	0
16	19.2	422,500	0.354	0.10	0
14	16.8	586,900	0.315	0.02	0
12	14.4	1,837,000	0.275	0.00	0
Tandem Axle					
52	62.4	21,320	0.484	1.14	1.9
48	57.6	42,870	0.452	0.10	0
44	52.8	124,900	0.420	0.15	0
40	48.0	372,900	0.387	0.26	0
36	43.2	885,800	0.354	0.28	0
32	38.4	930,700	0.319	0.08	0
28	33.6	1,656,000	0.284	0.01	0
Total				62.14	62.8

Note: 1 in. = 25.4 mm, 1 psi = 6.9 kpa, 1 pci = 271 kN/m³,
and 1 kip = 4.45 kN.

axle loads have a stress ratio less than 0.25 and a cracking index less than 0.01, they are not shown in the Table. In addition to the several design parameters shown in the table, a Young's modulus of 4×10^6 psi (27.6 Gpa) and a Poisson's ratio of 0.15 were assumed for the concrete. Unless specially noted, these values will be used throughout this paper.

The cracking index is based on the edge stress with the loading position shown in Figure 4. In view of the fact that most of the trucks have their outside wheel about 2 ft from the edge, PCA suggests the use of 90% of the edge stress for design purpose. Therefore, a stress adjustment factor of 0.9 was specified. It can be seen that both methods check quite well. The PCA method results in a cracking index of 62.8%, while the PMRPD computer program results in 62.14%. The fact that the PCA method gives a higher cracking index for heavier loads but a lower cracking index for lighter loads, as compared to the PMRPD method, is not surprising because the PMRPD method assumes a contact pressure of 80 psi (552 Kpa) for all loads, while the actual contact pressures should be greater for heavier loads. It is not known what contact pressures were used in the PCA method.

If greater contact pressures were used for heavier loads, as they should be, the more concentrated load with a smaller contact area would increase the stress and result in a greater cracking index. It is also not clear how the variation in concrete strength and the strength gain with age were handled in the PCA design charts.

Effect of Variability on Thickness Design

The fatigue analysis employed by PCA is deterministic in that a cracking index smaller than 100% is required without considering the variabilities of all the design factors. It is interesting to note that if the thickness of concrete is reduced to 9 in. (229 mm), the cracking index increases to 220%. To show the effect of variability of construction and materials on thickness design, three different levels of variability are assumed. The coefficients of variation for each design factor are presented in Table 2. Most of the data for medium variability were obtained from the results of AASHTO road test. The coefficients of variation for high variability are assumed twice as large as those for medium variability, while the coefficients of variation for low variability are only one half of the medium values.

Table 2. Coefficients of variations for three levels of variability

Variability	High	Medium	Low
Concrete Thickness	0.08	0.04	0.02
Axle Load	0.20	0.10	0.05
Concrete Modulus	0.20	0.10	0.05
Subgrade Modulus	0.70	0.35	0.175
Modulus of Rupture	0.20	0.10	0.05
Predicted Traffic	2.00	1.00	0.50

Table 3 shows the effect of variability on cracking index and the reliability of design for the example shown in Table 1. The reason that cracking index varies with the variability is due to the lighter loads with stress ratios smaller than 0.45, so the allowable number of repetitions depends on the probability that the stress ratio may exceed 0.45. The effect of variability on reliability is plotted in Figure 5. The reliability increases as the thickness increases or the variability decreases. If a reliability of 90% is specified, the thicknesses required are 10.5 in. (267 mm) for high variability, 10.1 in. (257 mm) for medium variability, and 9.7 in. (246 mm) for low variability. Table 4 shows the levels of reliability recommended by the AASHTO design guide (3).

The probabilistic method of design has several advantages over the deterministic method. Firstly, reliability is not so sensitive to thickness as the cracking index. A gradual increase in thickness should result in a gradual increase in reliability which makes sense. Secondly, reliability indicates the probability of success and has a physical meaning. A reliability of 90% implies that 10% of the pavement may fail at or before the end of the design period, while the cracking index does not relate even remotely to pavement conditions. Thirdly, pavement thickness should vary with the type and importance of highways by assigning different levels of reliability to each. It is just impractical to assign different cracking indices to different highways because the

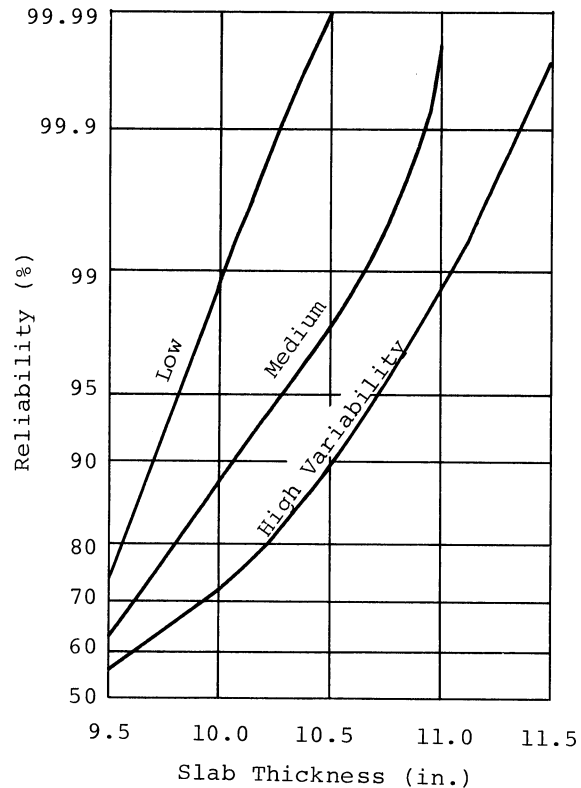


Figure 5. Effect of variability on thickness design under mixed traffic.
(1 in.=25.4 mm)

Table 3. Effect of variability on cracking index and reliability based on mixed traffic

Thickness (in.)	High Variability		Medium Variability		Low Variability	
	Cracking Index(%)	Reliability (%)	Cracking Index(%)	Reliability (%)	Cracking Index(%)	Reliability (%)
9.5	64.46	55.85	62.14	62.07	61.21	73.39
10.0	18.11	72.36	16.15	87.13	15.48	98.69
10.5	5.94	88.78	3.89	97.64	3.45	99.99
11.0	1.97	98.80	0.83	99.98	---	---
11.5	1.09	99.97	---	---	---	---

Note: 1 in. = 25.4 mm

Table 4. Levels of reliability recommended by AASHTO

Functional Classification	Urban	Rural
Interstate and Freeways	85-99.9	80-99.9
Principal Arterials	80-99	75-95
Collectors	80-95	75-95
Local	50-80	50-80

cracking index is very sensitive to thickness and varies over several orders of magnitude. Last but not least, the thickness required depends to a great extent on workmanship. Poorly constructed pavements with a wide range of variability should be thicker than well constructed pavements with good quality control.

Table 5 shows the effect of variability on cracking index and reliability based on 18 kip single axle loads. The same design parameters were used as in the previous example except that a subgrade modulus of 100 pci (27 Mpa) and an average daily traffic of 10 over a 20 year design period were assumed. As there is only one load, the cracking index is the same if the stress ratio is greater than 0.45. When the stress ratio is smaller than 0.45, the cracking index increases with an increase in variability. The effect of variability on reliability is plotted in Figure 6. It can be seen that the thickness of slab required depends on the level of variability. If a reliability of 99% is required, the slab thicknesses are 11.3 in. (287 mm) for high variability, 8.6 in. (218 mm) for medium variability, and 8.1 in. (206 mm) for low variability. Higher variability requires a much thicker slab when a high level of reliability is required.

Manual Method for Estimating Reliability

Given the coefficient of variation of each design factor, the reliability of the design for any given thickness can be easily obtained by the computer program. If the stress ratio and the cracking index are obtained by a

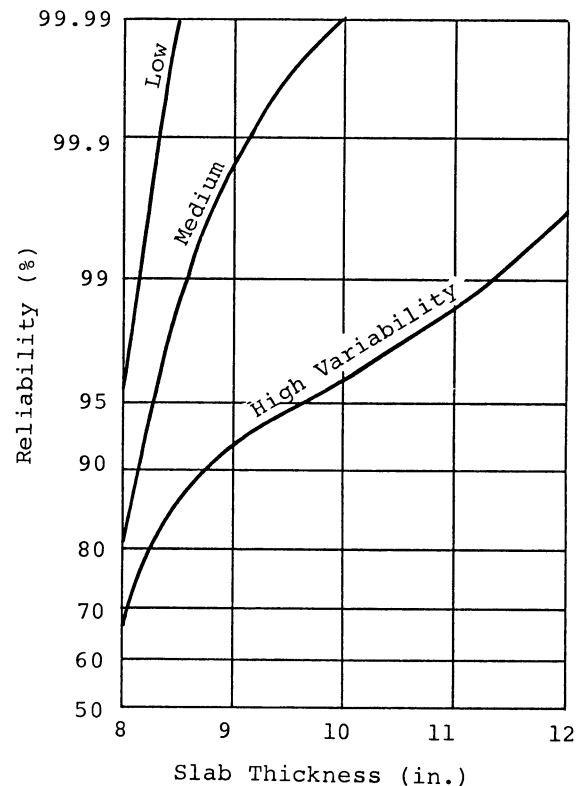


Figure 6. Effect of variability on thickness design under 18-kip single axle load. (1 in.=25.4 mm)

Table 5. Effect of Variability on cracking index and reliability based on 18 kip single axle load

Thickness (%)	Stress Ratio	High Variability		Medium Variability		Low Variability	
		Cracking Index(%)	Reliability (%)	Cracking Index(%)	Reliability (%)	Cracking Index(%)	Reliability (%)
8.0	0.500	9.58	66.36	9.58	80.08	9.58	95.44
8.5	0.458	0.38	86.00	0.38	98.46	0.38	100.00
9.0	0.421	0.10	92.52	0.08	99.85	—	—
9.5	0.389	0.08	94.42	0.05	99.97	—	—
10.0	0.361	0.06	96.07	0.02	100.00	—	—
11.0	0.314	0.03	98.48	—	—	—	—
12.0	0.275	0.01	99.65	—	—	—	—

Note: 1 in. = 25.4 mm and 1 kip = 4.45 kN

deterministic method, it is possible to estimate the coefficient of variation of the cracking index from that of each factor and thus ascertain the reliability of the design. The most difficult task is to determine the coefficient of variation of the maximum stress due to the variations of thickness, h , contact pressure, q , subgrade modulus, k , and Young's modulus of concrete, E_c . Once the coefficient of variation of stress is determined, the coefficient of variation of cracking index can be obtained by simple formulas.

Table 6 shows the coefficients of variation of maximum stress due to each of the factors mentioned above. If the coefficient of variation of thickness is assigned 1, while all other factors have no variations, the coefficient of variation of stress is 2.520 for single axle, 2.792 for tandem axles and 2.997 for tridem axles. It was found that the only factor which has a significant effect on the coefficient of variation of stress is the number of axles, while the effects of load magnitude, thickness, subgrade modulus, etc., are not significant. Note that the results of single axle are most consistent, while some variations exist for tandem and tridem axles.

The coefficient of variation of contact pressure is used to represent the variation of axle load. For a given axle load, the contact area varies with the contact pressure, so any change in contact pressure will cause a corresponding change in contact area. However, in the computer program the contact area is considered as deterministic with no variations. As the stress is directly proportional to the contact pressure, the coefficient of variation of stress should be equal to that of the contact pressure.

The variation of subgrade and concrete moduli has the same effect on the variation of stress. As indicated in Table 6, both result in a coefficient of variation of stress of 0.173 for single axle, 0.264 for tandem axles and 0.333 for tridem axles. It can be seen that the variation of slab thickness has most effect on the stress, while the variation of subgrade and concrete moduli has the least effect.

If all the four factors are assigned a coefficient of variation of 1, the coefficient of variation of stress is 2.722 for single axle, 2.989 for tandem axles and 3.195 for tridem axles. It can be easily shown that the coefficient of variation of stress due to all the factors can be obtained from each component by

$$C[s] = \left\{ (C[s]_h)^2 + (C[s]_q)^2 + (C[s]_k)^2 + (C[s]_{E_c})^2 \right\}^{\frac{1}{2}} \quad (29)$$

in which $C[s]$ = coefficient of variation of stress, and $C[s]_h$, $C[s]_q$, $C[s]_k$ and $C[s]_{E_c}$ = coefficient of variation of stress due to h , q , k and E_c , respectively. For single axle loads, Eq. 29 can be written as

$$C[s] = \left\{ 6.35(C[h])^2 + (C[q])^2 + 0.03(C[k])^2 + 0.03(C[E_c])^2 \right\}^{\frac{1}{2}} \quad (30)$$

Table 6. Ratio of coefficients of variation between maximum stress and several factors

Axle Load	Thickness (in.)	h	q	k	E_c	All
18 kip single	6	2.526	1.000	0.175	0.175	2.728
	8	2.520	1.000	0.173	0.173	2.722
	10	2.521	1.000	0.174	0.174	2.723
	12	2.512	1.000	0.171	0.171	2.714
	Avg	2.520	1.000	0.173	0.173	2.722
30 kip tandem	6	2.735	1.000	0.245	0.245	2.932
	8	2.806	1.000	0.269	0.269	3.003
	10	2.826	1.000	0.275	0.275	3.023
	12	2.801	1.000	0.267	0.267	2.999
	Avg	2.792	1.000	0.264	0.264	2.989
40 kip tridem	6	2.885	1.000	0.295	0.295	3.082
	8	3.039	1.000	0.347	0.352	3.237
	10	3.059	1.000	0.353	0.326	3.256
	12	3.006	1.000	0.335	0.335	3.203
	Avg	2.997	1.000	0.333	0.333	3.195

Note: 1 in. = 25.4 mm and 1 kip = 4.45 kN

in which $C[h]$, $C[q]$, $C[k]$ and $C[E_c]$ = coefficient of variation of h , q , k and E_c , respectively.

After $C[s]$ is found, the coefficient of variation of stress ratio $C[s/M]$ can be obtained by

$$C\left[\frac{s}{M}\right] = \left\{ (C[s])^2 + (C[M])^2 \right\}^{\frac{1}{2}} \quad (31)$$

in which $C[M]$ = coefficient of variation of concrete modulus of rupture. The coefficient of variation of allowable repetitions, $C[N]$, can be obtained from Eq. 24 by

$$C[N] = 27.81 \left(\frac{s}{M} \right) C\left[\frac{s}{M}\right] \quad (32)$$

The coefficient of variation of cracking index can be computed by

$$C[c] = \left\{ (C[N])^2 + (C[n])^2 \right\}^{\frac{1}{2}} \quad (33)$$

in which $C[n]$ = coefficient of variation of predicted repetitions. The standard deviation of $\log c$ is

$$\sigma_{\log c} = 0.4343 C[c] \quad (34)$$

Knowing $\log c$ and the standard deviation of $\log c$, the reliability of the design can be obtained from normal distribution tables.

The 8 in. (203 mm) slab with a stress ratio of 50.0% and a cracking index of 9.58%, as shown in Table 5, is used as an example. For high variability level, as shown in Table 2, $C[h] =$

0.08, $C[q] = 0.2$, $C[k] = 0.7$ and $C[E_c] = 0.2$,

from Eq. 30, $C[s] = \{6.35(0.08)^2 + (0.2)^2 + 0.03(0.7)^2 + 0.03(0.2)^2\}^{1/2} = 0.332$. Given

$C[M] = 0.2$, from Eq. 31 $C[s/M] = \{(0.332)^2 + (0.2)^2\}^{1/2} = 0.387$ and from Eq. 32 $C[N] = 27.81 \times 0.500 \times 0.387 = 5.38$. Given $C[n] = 2$, from

Eq. 33 $C[c] = \{(5.38)^2 + (2)^2\}^{1/2} = 5.74$ and

from Eq. 34 $\sigma_{\log c} = 0.4343 \times 5.74 = 2.49$. Given

$\log c = \log 0.0958 = -1.019$, the normal deviate for a cracking index of 1 is $(\log 1.0 + 1.019)/2.49 = 0.41$. The area under the normal curve from $-\infty$ to 0 is 0.659, so the reliability is 65.9%, which checks closely with the 66.36% obtained by PMRPD.

For medium variability with coefficients of variation one half of those for high variability, the standard deviation of $\log c$ is also one half, or $\sigma_{\log c} = 2.49/2 = 1.25$, and the

reliability is 79.4%, which checks with the 80.08% obtained from PMRPD. The same procedure can be applied to low variability with a reliability of 94.9%, which checks with the 95.94% obtained from PMRPD.

Comparison between Liquid and Solid Foundations

It is well known that the modulus of subgrade reaction, k , used with the liquid foundation is a fictitious property not characteristic of soil behaviors. However, due to simplicity in application, k values have been used most frequently for the design of concrete pavements. It will be useful if the modulus of subgrade reaction can be related to the elastic modulus and Poisson's ratio of the solid foundation, so that liquid foundations can be used to replace solid foundations, thus resulting in a large saving of computer time.

Vesic and Saxena (12) indicated that the value of k depends on the relative flexibility of the slab with respect to the foundation and that there is no single value of k that can give stresses and deflections in a concrete pavement comparable to those obtained by considering the foundation as an elastic solid. For computing stresses, they suggested the use of the following equation

$$k = \left(\frac{E_f}{E_c} \right)^{\frac{1}{3}} \frac{E_f}{(1+\nu_f^2)h} \quad (35)$$

in which E_f = elastic modulus of foundation and ν_f = Poisson's ratio of foundation. For computing deflections, they suggested that only 42% of the value obtained from Eq. 35 be used.

It was found in this study that Eq. 35 is applicable only to loads in the interior of the slab but not to loads near to the edge. For edge loading, a factor of 2.25 must be applied to obtain the same maximum stress, or

$$k = 2.25 \left[\frac{E_f}{E_c} \right]^{\frac{1}{3}} \frac{E_f}{(1+\nu_f^2)h} \quad (36)$$

Table 7 compares the maximum stresses together with the coefficients of variation between liquid and solid foundations for the case of medium variability. For the regular cases, the liquid foundation is the same as the cases presented in Table 5 with a k value of 100 pci (27.1 MN/m³). Based on Eq. 36 and an assumed ν_f of 0.4, the corresponding E_f is 4100 psi (28.3 Mpa) for 8 in. (203 mm) slab, 4500 psi (31.1 Mpa) for 9 in. (229 mm) slab, and 4850 psi (33.5 Mpa) for 10 in. (254 mm) slab. It can be seen that the maximum stresses for both foundations check very well. The coefficients of variation of stress also check very well if a

Table 7. Comparison between liquid and solid foundations

(a) Regular Cases

Thickness (in.)	8		9		10	
Foundation Type	Liquid	Solid	Liquid	Solid	Liquid	Solid
Foundation Modulus*	100.0	4100.0	100.0	4500.0	100.0	4850.0
C.V. of Modulus (%)	35.0	20.0	35.0	20.0	35.0	20.0
Maximum Stress (%)	324.9	324.1	273.7	272.5	234.6	233.7
C.V. of Stress (%)	15.7	15.7	15.7	15.7	15.6	15.6
Stress (psi) based on AASHTO Correlation	284.0 (k = 211 pci)		234.4 (k = 232 pci)		198.8 (k = 250 pci)	

(b) Extreme Cases

Thickness (in.)	6		6		12		12	
Concrete Modulus (psi)	2x10 ⁶		8x10 ⁶		2x10 ⁶		8x10 ⁶	
Foundation Type	Liquid	Solid	Liquid	Solid	Liquid	Solid	Liquid	Solid
Foundation Modulus*	950	15000	27.8	1500	475	15000	13.9	1500
C.V. of Modulus (%)	35.0	20.0	35.0	20.0	35.0	20.0	35.0	20.0
Maximum Stress (psi)	283.4	266.5	703.1	701.1	119.9	119.3	228.3	246.9
C.V. of Stress (%)	16.6	17.2	15.4	15.5	15.6	15.9	13.1	13.7
Stress (psi) based on AASHTO correlation	296.1 (k=773 pci)		586.8 (k=77.3 pci)		110.0 (k=773 pci)		203.9 (k=77.3 pci)	

Note: * units for liquid foundation in pci and solid in psi

1 in. = 25.4 mm, 1 psi = 6.9 kpa, 1 pci = 271 kN/m³

coefficient of variation of 0.2 is used for E_s and of 0.35 for k . The reason that a larger coefficient should be used for the liquid foundation to match with the solid foundation is because the liquid foundation is a diagonal matrix with all non-diagonal elements equal to zero whereas the solid foundation is more fully populated. The use of a smaller coefficient of variation for E_s is reasonable because E_s can be more accurately measured compared to k .

The regular cases assume a thickness from 8 to 10 in. (203 to 254 mm) and a Young's modulus of 4×10^6 psi (27.6 Gpa) for concrete. Even for the extreme cases with a thickness of 6 and 12 in. (152 to 305 mm) and a concrete modulus of 2×10^6 and 8×10^6 psi (13.8 and 55.2 Gpa), the discrepancy between the two types of foundation is still quite small, indicating the validity of Eq. 36. Even in the two extreme cases with k values of 950 and 13.9 pci (257 and 3.8 MN/m³), the differences are less than 10%. Note that the average coefficient of variation of stress is about 15.4%. However, when the slab thickness and concrete modulus are extremely small, the coefficient of variation of stress becomes larger and when they are extremely large, the coefficient becomes smaller. This trend appears to be reasonable.

Also shown in Table 7 are the stresses based on the AASHTO equation that

$$k = \frac{E_s}{19.4} \quad (37)$$

It can be seen that the results of liquid foundations do not check well with those of solid foundations if Eq. 37 is used.

Comparison between Solid and Layer Foundations

Figure 7 shows a concrete slab on a two-layer foundation. The top layer has a thickness of 6 in. (152 mm) and an elastic modulus of 15,000 psi (104 Mpa); the bottom layer is of infinite thickness and has an elastic modulus of 3,000 psi (20.7 Mpa). A Poisson's ratio of 0.4 is assumed for both layers.

Table 8 compares the maximum stresses together with the coefficients of variation between solid and layer foundations. To match the maximum stresses in the slab on layer foundation, the solid foundation should have an elastic modulus of 3350 psi (23.1 Mpa) for 8 in. (203 mm) slab, 3300 psi (22.8 Mpa) for 9 in. (229 mm) slab, and 3250 psi (22.4 Mpa) for 10 in. (254 mm) slab. Although the equivalent E_s decreases slightly with the increase in slab thickness, the decrease is so small and can be ignored. This is reasonable because the stiffness matrices of solid and layer foundations are both fully populated in contrast to the diagonal matrix for liquid foundation.

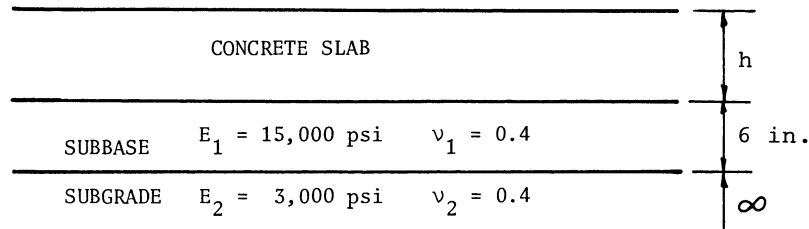


Figure 7. Slab on a layered foundation.
(1 in.=25.4 mm, 1 psi=6.9 kPa)

Table 8. Comparison between solid and layer foundations

Thickness (in.)	8		9		10	
Foundation Type	Solid	Layer	Solid	Layer	Solid	Layer
Foundation Modulus (psi)	3350.0	Fig.7	3300.0	Fig.3	3250.0	Fig.7
C.V. of Modulus (%)	20.0	20.0	20.0	20.0	20.0	20.0
Maximum Stress (psi)	339.1	339.3	292.1	292.1	254.5	254.7
C.V. of Stress (%)	15.7	15.8	15.6	15.7	15.4	15.5
Stress (psi) based on AASHTO correlation	295.7 (k=170 pci)		248.4 (k=170 pci)		212.5 (k=170 pci)	

Note: 1 in. = 25.4 mm, 1 psi = 6.9 kpa and 1 pci = 271 kN/m³

In the layer foundation, the coefficient of variation of the foundation modulus is applied only to the first layer or subbase, while the other layer or subgrade is assumed deterministic with no variations. It is possible to consider the variations of each layer, but the computer program will become much more complicated. The only effect of layer foundation on the slab is the deflection on the surface of foundation. For a given modulus ratio between the layers, the deflection is inversely proportional to the modulus of the first layer. It is not only more convenient but also more practical to estimate or measure the coefficient of variation of foundation deflections and use it as the coefficient of variation of the elastic modulus for the first layer.

Also shown in Table 8 are the stresses based on liquid foundation with a k value of 170 pci (46.1 MN/m³), which was obtained from the AASHTO correlation chart (3). It can be seen that the stresses based on the AASHTO correlation do not check well with the layer foundation.

CONCLUSIONS

A probabilistic finite element computer program named PMRPD was developed for the practical design of concrete pavements. The design is based on the edge stress and the cumulative fatigue concept. From the variability of various design factors, the cracking index can be related directly to the reliability, so the effect of variability on thickness design can be evaluated. It was demonstrated that the use of reliability as a design criterion, instead of the deterministic cracking index, resulted in a more reasonable and realistic design.

Based on the results of this study, the following conclusions can be drawn:

1. A probabilistic procedure can be incorporated into a deterministic finite element computer program using the first order term in a Taylor series expansion. With due considerations to the dependence or independence

of random variables, the procedure yields reasonable results even a large number of simultaneous equations are involved.

2. Cracking index is very sensitive to the thickness of slab. Current practice of thickness design based solely on cracking index without considering the variability of design factors; such as quality of construction and material tolerance in construction, etc., is unsound and results in unrealistic design. The thickness required should increase as the variability increases.

3. Given the stress ratio, the cracking index and the variability of various design factors, the reliability of the design can be estimated by a manual method, which checks very closely with the computer solutions.

4. The relationship between the k value in a liquid foundation and the E_s value in a solid foundation is not unique and depends on the design criterion. Designs based on interior loading have a different relationship compared to those based on edge loading. An equation relating k to E_s for the case of edge loading was presented. For a given E_s , the equivalent k value decreases with the increase in slab thickness. To obtain the same coefficient of variation of stress and reliability, the coefficient of variation used for k should be much greater than that for E_s .

5. To replace a layer foundation by a solid foundation, a single equivalent E_s can be found, which is practically independent of slab thickness. If the same coefficient of variation is assumed for both foundations, the same coefficient of variation of stress will be obtained.

6. The correlations between k and E_s and between k and the elastic moduli of layer foundations, as presented in the 1986 AASHTO design guide, are not applicable to the case where the design is based on the edge stress. This is not surprising because the AASHTO design method is based on an empirical equation instead of the edge stress.

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APPENDIX

An important part of the computer program is to solve Eq. 5 and determine the values and the variances of $\{\delta'\}$. In the Gauss elimination method, the stiffness matrix is reduced to an upper triangular matrix and δ'_n is determined from the last or n th equation. By back substituting δ'_n into the $(n-1)$ th equation, δ'_{n-1} is then determined. The process is repeated until all $\{\delta'\}$ are determined. In determining the variance by back substitution, it is necessary to use $D[\delta']$, which is the variance of δ' before square, so the appropriate positive or negative sign can be preserved. The following example is given to illustrate the procedure.

Given the simultaneous equation similar to Eq. 5

$$\begin{bmatrix} 1+r & 3r \\ 3r & 3+r \end{bmatrix} \begin{bmatrix} \delta'_1 \\ \delta'_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad (38)$$

with the random variable $r=1$ and the variance $V[r] = 0.09$, determine the values of δ'_1 and δ'_2 and their variances.

Due to the simplicity of the equations with only two unknowns, δ'_1 and δ'_2 , the closed-form solutions can be easily obtained. These solutions will then be used to check those obtained by the Gauss elimination method.

It can be easily shown that the solutions of Eq. 38 are

$$\delta'_1 = \frac{16r - 15}{8r^2 - 4r - 3} \quad (39)$$

$$\delta'_2 = \frac{8r - 7}{8r^2 - 4r - 3} \quad (40)$$

With $r=1$, from Eq. 39, $\delta'_1 = 1$ and from Eq. 40, $\delta'_2 = 1$. The variance of δ'_1 can be determined from Eqs. 7 and 39.

$$V[\delta'_1] = \left\{ \frac{(8r^2 - 4r - 3)(16) - (16r - 15)(16r - 4)}{(8r^2 - 4r - 3)^2} \right\} V[r] \quad (41)$$

with $r=1$ and $V[r] = 0.09$, from Eq. 41, $V[\delta'_1] = (16-12)^2 \times 0.09 = 1.44$. From Eqs. 7 and 40

$$V[\delta'_2] = \left\{ \frac{(8r^2 - 4r - 3)(8) - (8r - 7)(16r - 4)}{(8r^2 - 4r - 3)^2} \right\} V[r] \quad (42)$$

From Eq. 42, $V[\delta'_2] = (8-12)^2 \times 0.09 = 1.44$.

Although both δ'_1 and δ'_2 have the same variance of 1.44. The variances before square have opposite sign with $D[\delta'_1] = +1.2$ and $D[\delta'_2] = -1.2$. The correct sign should be used in back substitution.

In applying the Gauss elimination method, the original matrix and the value of D , as enclosed in parenthesis, can be determined from Eq. 38 with $r=1$ and $D[r] = \sqrt{V[r]} = 0.3$, or

$$\begin{bmatrix} c=2 & b=3 \\ (D[c]=0.3) & (D[b]=0.9) \end{bmatrix} \begin{bmatrix} \delta'_1 \\ \delta'_2 \end{bmatrix} = \begin{bmatrix} b=5 \\ (D[b]=0) \end{bmatrix} \quad (43)$$

The use of b and c in Eq. 43 is for identification purpose. When the first equation is divided by c , from Eq. 14a, the D values are $0.9/2 - 3 \times 0.3/4 = 0.225$ and $0/2 - 5 \times 0.3/4 = -0.375$, so Eq. 43 becomes

$$\begin{bmatrix} 1 & d=1.5 \\ (0) & (D[d]=0.225) \\ c=3 & b=4 \\ (D[c]=0.9) & (D[b]=0.3) \end{bmatrix} \begin{bmatrix} \delta'_1 \\ \delta'_2 \end{bmatrix} = \begin{bmatrix} d=2.5 \\ (D[d]=-0.375) \\ b=7 \\ (D[b]=0) \end{bmatrix} \quad (44)$$

When the first equation is multiplied by c and then subtracted from the second equation, as is the general procedure used in Gauss elimination, from Eq. 14b, the D values are $0.3 - 1.5 \times 0.9 - 3 \times 0.225 = -1.725$ and $0 - 2.5 \times 0.9 - 3(-0.375) = -1.125$, so Eq. 44 is reduced to

$$\begin{bmatrix} 1 & 1.5 \\ (0) & (0.225) \\ 0 & c = -0.5 \\ (0) & (D[c]=-1.725) \end{bmatrix} \begin{bmatrix} \delta'_1 \\ \delta'_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ (-0.375) \\ b = -0.5 \\ (D[b]=-1.125) \end{bmatrix} \quad (45)$$

From the second equation in Eq. 43, $\delta'_2 = b/c = (-0.5)/(-0.5) = 1$ and from Eq. 14a $D[\delta'_2] =$

$-1.125/(-0.5) - (-0.5)(-1.725)/0.25 = -1.2$, which checks with the closed-form solution. The values of δ'_1 and $D[\delta'_1]$ can be determined

from the first equation in Eq. 45, or $\delta'_1 = 2.5 - 1.5 \delta'_2 = 1$. From Eq. 14b, $D[\delta'_1] = D[2.5] - \delta'_2 D[1.5] - 1.5 D[\delta'_2] = -0.375 - 1 \times 0.225 - 1.5 \times (-1.2) = 1.2$, which also checks with the closed-form solution.

The same procedure applies when a large number of equations are to be solved. The first equation is used to eliminate the first unknown and the i th equation for the i th unknown. Finally, the last equation has only one unknown and can be solved. By back substitution, the remaining unknowns can be determined.