

A RIGOROUS ANALYTICAL SOLUTION OF A CONCRETE SLAB
SUBMITTED TO INTERIOR AND EDGE LOADS
WITH NO, PARTIAL AND FULL SHEAR TRANSFER AT THE EDGE

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Introduction

During last years, many authors have tried to demonstrate the well known Westergaard's formulas for slabs on an elastic subgrade. While they succeeded for interior loading, nobody seems to have been able to prove the formulas for edge loading.

Comparisons have been made using finite element methods and discrepancies in minus and in plus with the Westergaard's formulas were discovered.

In this paper the general analytical solution for both interior and edge loading

is developed; moreover, partial and full shear transfer at the edge can be taken into account. The presented solution in fact allows to determine any stress or deflection at any distance from an interior or an edge load.

A comparison with the classical Westergaard's relations shows that:

- The original formula from 1926 for the deflection at the edge is fairly correct;

- The original formula from 1926 for the stress at the edge is not correct;

- The new formula from 1948 for the stress at the edge is correct.

1. General solution for an infinite slab submitted to a rectangular load.

Consider the slab of height h represented at figure 1 (in cartesian coordinates), submitted to a uniformly distributed vertical load p over a rectangular area with length $2a$, width $2b$ and resting on an elastic subgrade.

The subgrade reaction is given by q .

The basic equations for this problem are [Timoshenko (1)]

$$M_x = -D \left[\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right] \quad (1)$$

$$M_y = -D \left[\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right] \quad (2)$$

$$M_{xy} = -D (1 - \mu) \frac{\partial^2 w}{\partial x \partial y} \quad (3)$$

$$Q_x = -D \frac{\partial}{\partial x} \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \quad (4)$$

$$Q_y = -D \frac{\partial}{\partial y} \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \quad (5)$$

$$\frac{p - q}{D} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \quad (6)$$

where M_x is the flexural moment in the direction x

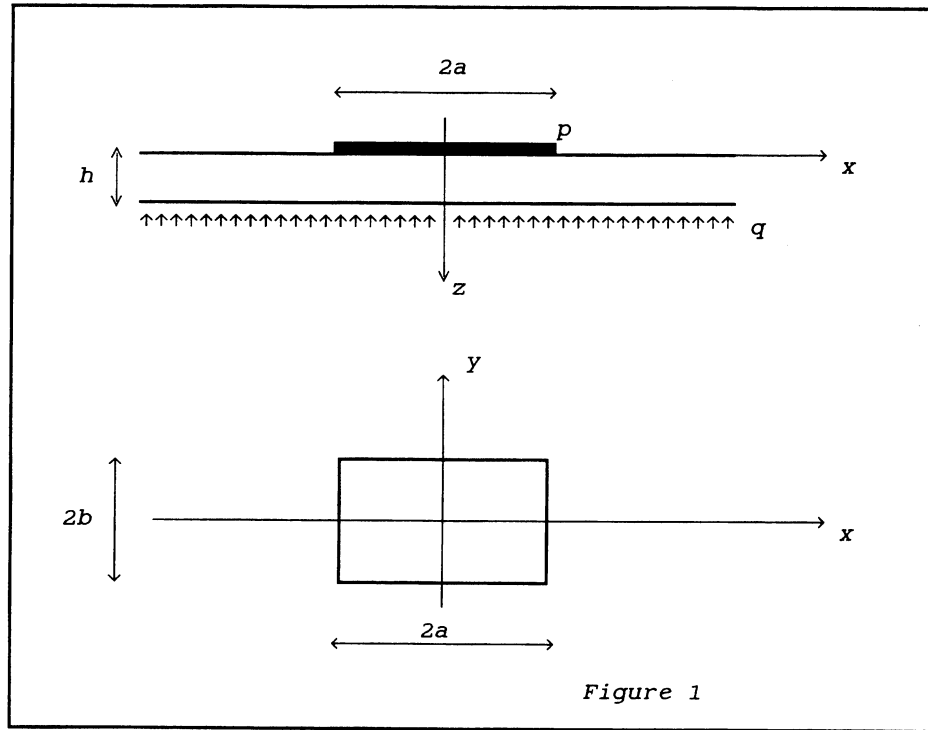


Figure 1

M_y is the flexural moment in the direction y

M_{xy} is the moment of torsion in the plane xy

Q_x is the shearforce in the plane yz

Q_y is the shearforce in the plane xz

$D = Eh^3/[12(1-\mu^2)]$

E is the modulus of elasticity of the slab

μ is Poisson's ratio of the slab

q is the subgrade reaction

w is the deflection at a point x, y

If we make the assumption that $q = k.w$, with k the stiffness modulus of the subgrade, (6) transforms into

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] + k \frac{w}{D} = \frac{p}{D} \quad (7)$$

The general solution for (7) is given by

$$p = \frac{4p}{\pi^2} \int_0^\infty \int_0^\infty \frac{\cos(tx/l) \sin(ta/l) \cos(sy/l) \sin(sb/l)}{ts} dt ds$$

$$= \begin{cases} p & \text{for } |x| < a \text{ and } |y| < b \\ 0 & \text{for } |x| > a \text{ or } |y| > b \end{cases} \quad (8)$$

$$w = \int_0^\infty \int_0^\infty C(s, t) \frac{\cos(tx/l) \sin(ta/l) \cos(sy/l) \sin(sb/l)}{ts} dt ds \quad (9)$$

Writing (8) and (9) in (7) yields

$$C(s, t) = \frac{4p}{\pi^2 D} \frac{l^4}{(t^4 + 2t^2 s^2 + s^4) + 1} \quad (10)$$

with $l^4 = D/k$.

Hence

$$w = \frac{4p}{\pi^2 k} \int_0^\infty \int_0^\infty \frac{\cos(tx/l) \sin(ta/l) \cos(sy/l) \sin(sb/l)}{ts (t^4 + 2t^2 s^2 + s^4 + 1)} dt ds \quad (11)$$

A partial solution of (11) can be obtained analytically (the integration is performed in appendix 1) and is given by

For $0 \leq x \leq a$

$$w = \frac{p}{\pi k} \int_0^\infty \frac{\cos(sy/l) \sin(sb/l)}{s (s^4 + 1)} \left\{ 2 - e^{-\alpha(a-x)/l} \left[\cos \frac{x-a}{l} \beta - s^2 \sin \frac{x-a}{l} \beta \right] - e^{-\alpha(a+x)/l} \left[\cos \frac{x+a}{l} \beta + s^2 \sin \frac{x+a}{l} \beta \right] \right\} ds \quad (12)$$

For $x > a$

$$w = \frac{p}{\pi k} \int_0^\infty \frac{\cos(sy/l) \sin(sb/l)}{s (s^4 + 1)} \left\{ e^{-\alpha(x-a)/l} \left[\cos \frac{x-a}{l} \beta + s^2 \sin \frac{x-a}{l} \beta \right] - e^{-\alpha(a+x)/l} \left[\cos \frac{x+a}{l} \beta + s^2 \sin \frac{x+a}{l} \beta \right] \right\} ds \quad (13)$$

where

$$\alpha = \left[\frac{(s^4 + 1)^{1/2} + s^2}{2} \right]^{1/2} \quad \text{and} \quad \beta = \left[\frac{(s^4 + 1)^{1/2} - s^2}{2} \right]^{1/2} \quad (14)$$

As will be seen further, the partial solution given by (12) and (13) is absolutely required for the establishment of the solution for a limited slab.

2. Slab submitted to an isolated load

Relation (11) can be transformed for an isolated load P by writing

$$P = \lim_{\substack{a \rightarrow 0 \\ b \rightarrow 0}} 4p \ a \ b$$

Hence

$$\lim_{\substack{a \rightarrow 0 \\ b \rightarrow 0}} 4p \frac{\sin(at/l) \sin(bs/l)}{ts} = \lim_{\substack{a \rightarrow 0 \\ b \rightarrow 0}} \frac{P}{a \ b} \frac{\sin(at/l) \sin(bs/l)}{ts} = \frac{P}{l^2}$$

$$w = \frac{P}{k\pi^2 l^2} \int_0^\infty \int_0^\infty \frac{\cos(tx/l)\cos(sy/l)}{t^4 + 2t^2 s^2 + s^4 + 1} dt ds \quad (15)$$

In the axis of the load ($x = 0, y = 0$), (15) reduces to

$$w = \frac{P}{k\pi^2 l^2} \int_0^\infty \int_0^\infty \frac{1}{t^4 + 2t^2 s^2 + s^4 + 1} dt ds \quad (16)$$

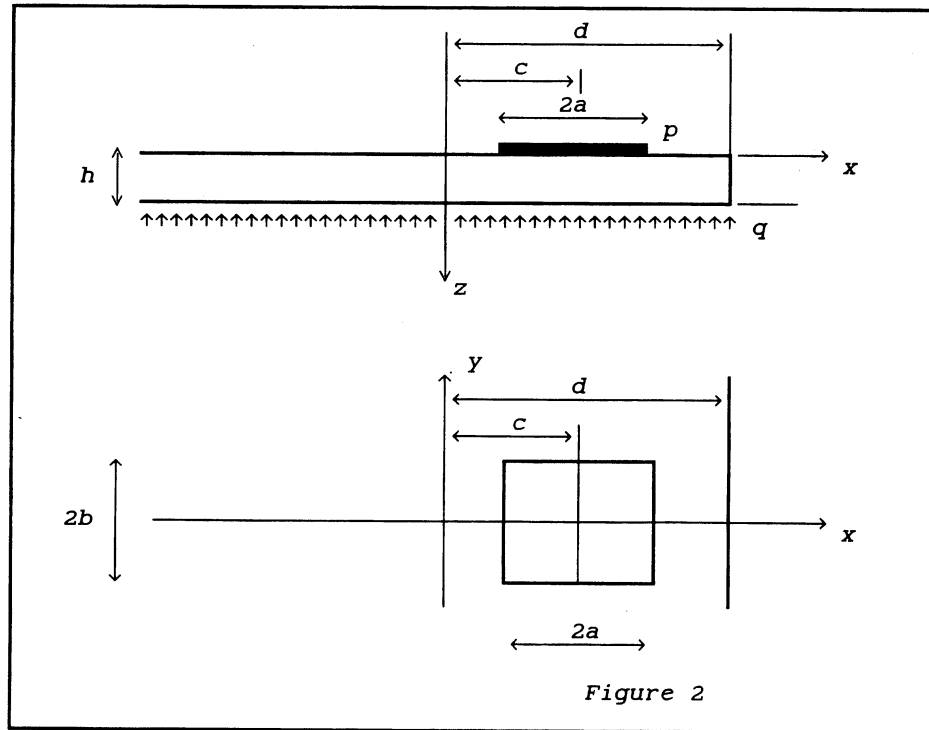
The resolution of the integral is given in Appendix 2 and is written

$$w = \frac{P}{8 kl^2} \quad (17)$$

which is the well known formula for the deflection of an infinite slab submitted to an isolated load.

3. Slab of finite dimensions

Consider a slab with a finite boundary in $x = d$ as represented at figure 2.



The boundary conditions for a slab limited in $x = d$ are

$$M_x(x=d) = 0, \quad M_{xy}(x=d) = 0, \quad Q_x(x=d) = 0 \quad (18)$$

However it can be shown, [Timoshenko (1)], that the three conditions given by (18) can be reduced to only 2 conditions:

$$\left[\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right] = 0 \quad (19)$$

$$\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu) \frac{\partial^3 w}{\partial x \partial y^2} \right] = 0 \quad (20)$$

These boundary conditions can be expressed by 2 independant solutions of the homogeneous equation (21):

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] + k \frac{w}{D} = 0 \quad (21)$$

Those solutions, derived in Appendix 3, are given by

$$w_A = \frac{p}{\pi k} \int_0^\infty A(s) e^{\alpha x/l} \cos(\beta x/l) \cos(sy/l) \sin(sb/l) ds \quad (22)$$

$$w_B = \frac{p}{\pi k} \int_0^\infty B(s) e^{\alpha x/l} \sin(\beta x/l) \cos(sy/l) \sin(sb/l) ds \quad (23)$$

wherein α and β are given by (14) and $A(s)$ and $B(s)$ are determined through the boundary conditions. The general solution is written

$$w = w_1 + w_A + w_B \quad (24)$$

where w_1 , the deflection for the infinite slab is given by (12) or (13).

The boundary conditions become

$$\left[\frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right] [w_1 + w_A + w_B] = 0 \quad (25)$$

$$\left[\frac{\partial^3}{\partial x^3} + (2-\mu) \frac{\partial^3}{\partial x \partial y^2} \right] [w_1 + w_A + w_B] = 0 \quad (26)$$

Shear transfer at the boundary can be taken into account by writing boundary condition (26) as follows

$$\left[\frac{\partial^3}{\partial x^3} + (2-\mu) \frac{\partial^3}{\partial x \partial y^2} \right] [w_1 + w_A + w_B] = \gamma \left[\frac{\partial^3}{\partial x^3} + (2-\mu) \frac{\partial^3}{\partial x \partial y^2} \right] [w_1] \quad (27)$$

The value of γ is related to the amount of shear transfer:

if $\gamma = 1$, one has full shear (or load) transfer;

if $0 \leq \gamma \leq 1$, one has partial shear transfer;

if $\gamma = 0$, one has no shear transfer.

4. Determination of the functions $A(s)$ and $B(s)$

The determination of the relations expressing $A(s)$ and $B(s)$ first requires the establishment of a series of derivatives which are derived in Appendix 4.

Further, considering the slab represented at figure 2, we must express boundary conditions (25) and (27) for $x = d$. The detailed computations are given in Appendix 5.

The resolution of the system of boundary conditions yields

$$A(s)e^{\alpha d/l} = - \frac{[F_3 \cos(\beta d/l) + F_4 \sin(\beta d/l)]}{\Delta} \quad (28)$$

$$B(s)e^{\alpha d/l} = \frac{[F_4 \cos(\beta d/l) - F_3 \sin(\beta d/l)]}{\Delta} \quad (29)$$

where

$$\Delta = 2\alpha s^2(1-\mu) + \beta[1 - (1-\mu)^2 s^4]$$

$$F_3 = [F_1[(1-\mu)\beta s^2 - \alpha] + F_2]$$

$$F_4 = [F_1[(1-\mu)\alpha s^2 + \beta] + F_2(1-\mu)s^2]$$

$$F_1 = \frac{1}{s} \left[e^{-(d-c+a)\alpha/l} \sin \frac{d-c+a}{l} \beta - e^{-(d-c-a)\alpha/l} \sin \frac{d-c-a}{l} \beta \right] + \frac{s\mu}{(s^4 + 1)} \left\{ e^{-(d-c-a)\alpha/l} \left[\cos \frac{d-c-a}{l} \beta + s^2 \sin \frac{d-c-a}{l} \beta \right] - e^{-(d-c+a)\alpha/l} \left[\cos \frac{d-c+a}{l} \beta + s^2 \sin \frac{d-c+a}{l} \beta \right] \right\}$$

$$F_2 = - (1-\gamma) \left\{ \frac{1}{s} e^{-(d-c-a)\alpha/l} \left[\beta \cos \frac{d-c-a}{l} \beta - \alpha \sin \frac{d-c-a}{l} \beta \right] - \frac{1}{s} e^{-(d-c+a)\alpha/l} \left[\beta \cos \frac{d-c+a}{l} \beta - \alpha \sin \frac{d-c+a}{l} \beta \right] + \frac{s(2-\mu)}{(s^4 + 1)} \times \left[e^{-(d-c-a)\alpha/l} \left[(\alpha - \beta s^2) \cos \frac{d-c-a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{d-c-a}{l} \beta \right] - e^{-(d-c+a)\alpha/l} \left[(\alpha - \beta s^2) \cos \frac{d-c+a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{d-c+a}{l} \beta \right] \right] \right\}$$

5. The stresses and deflection in a finite slab

Consider a slab with a boundary for $x = d$ submitted to a load which axis coordinates are $(c, 0)$ as represented at figure 2.

The deflection for any point (x, y) is given by

$$w(x, y) = w_i(x-c, y) + \frac{p}{\pi k} \int_0^{\infty} [A(s)e^{\alpha x/l} \cos(\beta x/l) + B(s)e^{\alpha x/l} \sin(\beta x/l)] \cos(sy/l) \sin(sb/l) ds$$

Replacing $A(s)$ and $B(s)$ by their values of (28) and (29) yields

$$w(x, y) = w_i(x-c, y) - \frac{p}{\pi k} \int_0^{\infty} \left[F_3 \cos[(d-x)\beta/l] + F_4 \sin[(d-x)\beta/l] \right] \frac{e^{-\alpha(d-x)/l} \cos(sy/l) \sin(sb/l)}{\Delta} ds \quad (30)$$

The normal stress σ_x is given by

$$\begin{aligned} \sigma_x &= -\frac{6D}{h^2} \left[\frac{d^2 w}{dx^2} + \mu \frac{d^2 w}{dy^2} \right] \\ \sigma_x &= \sigma_{xi} - \frac{6pD}{\pi k l^2 h^2} \int_0^{\infty} \left[A(s)e^{\alpha x/l} [s^2 \cos(x\beta/l) - \sin(x\beta/l)] + B(s)e^{\alpha x/l} [s^2 \sin(x\beta/l) + \cos(x\beta/l)] \right] \cos(sy/l) \sin(sb/l) ds + \\ &+ \frac{6\mu pD}{\pi k l^2 h^2} \int_0^{\infty} s^2 e^{\alpha x/l} [A(s) \cos(x\beta/l) + B(s) \sin(x\beta/l)] \cos(sy/l) \sin(sb/l) ds \\ \sigma_x &= \sigma_{xi} + \frac{6pD}{\pi k l^2 h^2} \int_0^{\infty} \left[[F_3(1-\mu)s^2 - F_4] \cos[(d-x)\beta/l] + [F_4(1-\mu)s^2 + F_3] \sin[(d-x)\beta/l] \right] \frac{e^{-\alpha(d-x)/l} \cos(sy/l) \sin(bs/l)}{\Delta} ds \end{aligned} \quad (31)$$

The normal stress σ_y is given by

$$\begin{aligned} \sigma_y &= -\frac{6D}{h^2} \left[\mu \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right] \\ \sigma_y &= \sigma_{yi} - \frac{6\mu pD}{\pi k l^2 h^2} \int_0^{\infty} \left[A(s)e^{\alpha x/l} [s^2 \cos(x\beta/l) - \sin(x\beta/l)] + B(s)e^{\alpha x/l} [s^2 \sin(x\beta/l) + \cos(x\beta/l)] \right] \cos(sy/l) \sin(sb/l) ds + \\ &+ \frac{6pD}{\pi k l^2 h^2} \int_0^{\infty} s^2 e^{\alpha x/l} [A(s) \cos(x\beta/l) + B(s) \sin(x\beta/l)] \sin(sb/l) ds \end{aligned}$$

$$\sigma_y = \sigma_{yi} - \frac{6pD}{\pi k l^2 h^2} \int_0^\infty \left[[F_3(1-\mu)s^2 + \mu F_4] \cos[(d-x)\beta/l] + [F_4(1-\mu)s^2 - \mu F_3] \sin[(d-x)\beta/l] \right] \frac{e^{-\alpha(d-x)/l} \cos(sy/l) \sin(bs/l)}{\Delta} ds \quad (32)$$

The shear stress τ_{xy} is given by

$$\begin{aligned} \tau_{xy} &= - \frac{6D}{h^2} \frac{d^2 w}{dx dy} (1+\mu) \\ \tau_{xy} &= \tau_{xyi} + \frac{6pD}{\pi k l^2 h^2} \int_0^\infty \left[sA(s)e^{\alpha x/l} [\alpha \cos(x\beta/l) - \beta \sin(x\beta/l)] + sB(s)e^{\alpha x/l} [\alpha \sin(x\beta/l) + \beta \cos(x\beta/l)] \right] (1+\mu) \sin(sy/l) \sin(sb/l) ds \\ \tau_{xy} &= \tau_{xyi} - \frac{6pD}{\pi k l^2 h^2} \int_0^\infty \left\{ F_3 [\alpha \cos[(d-x)\beta/l] + \beta \sin[(d-x)\beta/l]] - F_4 [\beta \cos[(d-x)\beta/l] - \alpha \sin[(d-x)\beta/l]] \right\} \times \\ &\quad \times (1+\mu) \frac{se^{-\alpha(d-x)/l} \sin(sy/l) \sin(sb/l)}{\Delta} ds \quad (33) \end{aligned}$$

The stresses can be obtained for any system of coordinates (x', y') making an angle ϑ with the original system (x, y) , using the well known relations

$$\begin{aligned} \sigma_{x'} &= \sigma_x \cos^2 \vartheta + \sigma_y \sin^2 \vartheta + 2\tau_{xy} \cos \vartheta \sin \vartheta \\ \sigma_{y'} &= \sigma_x \sin^2 \vartheta + \sigma_y \cos^2 \vartheta - 2\tau_{xy} \cos \vartheta \sin \vartheta \\ \tau_{x'y'} &= (\sigma_y - \sigma_x) \cos \vartheta \sin \vartheta + \tau_{xy} (\cos^2 \vartheta - \sin^2 \vartheta) \end{aligned} \quad (34)$$

6. Numerical applications

Consider a rectangular slab on an elastic subgrade with $k = 100 \text{ MN/m}^3$, and a joint at $x = d \text{ m}$; the formal value of d is not required.

The slab is $0,20 \text{ m}$ thick, its radius of relative stiffness, $l = 1$, and its Poisson's ratio $\mu = 0,20$.

Consider a load of $p = 1 \text{ MN/m}^2$ uniformly

distributed over a square area of $0,50 \times 0,50 \text{ m}^2$.

6.1. Slab of infinite extend

The deflection in the axis of the load is given by (11) or (12)

with $x = y = 0$:

$$w_i = 3.017 \cdot 10^{-4} \text{ m}$$

The normal stress, $\sigma_x = \sigma_y$, is given by (31) or (32) wherein $A(s)$ and $B(s)$ are taken zero and $x = y = 0$:

$$\sigma_x = \sigma_y = 6.73 \text{ MN/m}^2$$

The comparison with Westergaard's results is performed using the equations proposed by Ioannides et al [6].

The deflection in the axis of a circular load is given by equation (I-7) of [6]:

$$w_i = \frac{P}{8kl^2} \left[1 + \frac{1}{2\pi} [\ln(a/2l) - 0.673] (a/l)^2 \right]$$

The radius of a circular load corresponding with a square load of 0.25 m^2 is $a = 0.282 \text{ m}$.

Hence

$$w_i = 3.02 \cdot 10^{-4} \text{ m}$$

The flexural stress is given by equation (I-3) of [6]:

$$\sigma_x = \sigma_y = \frac{3P(1+\mu)}{2\pi h^2} [\ln(2l/c') - 0.077]$$

where $c' = [e^{(\pi/4 - 1)} \cdot c] / \sqrt{2}$ and c is the side length of the load ($c = 2a$).

Hence

$$\sigma_x = \sigma_y = 6.70 \text{ MN/m}^2$$

6.2. Finite slab with an edge load

We compute deflection and normal stress σ_y at the edge in the middle of the load.

The deflection is given by (30) with $x = d$, $y = 0$, $c = d - a$:

$$w = w_i + w_s$$

where w_i is the solution for the infinite slab and w_s , the correction for the finite slab.

This correction is a function of the shear transfer.

With no shear transfer ($\gamma = 0$):

$$w = 2.91 \cdot 10^{-4} + 6.02 \cdot 10^{-4} = 8.93 \cdot 10^{-4} \text{ m}$$

With half shear transfer ($\gamma = 0.5$):

$$w = 2.91 \cdot 10^{-4} + 3.73 \cdot 10^{-4} = 6.64 \cdot 10^{-4} \text{ m}$$

With full shear transfer ($\gamma = 1$):

$$w = 2.91 \cdot 10^{-4} + 1.50 \cdot 10^{-4} = 4.41 \cdot 10^{-4} \text{ m}$$

The normal stress σ_y , parallel with the edge is given by (32):

With no shear transfer ($\gamma = 0$):

$$\sigma_y = 5.72 + 6.58 = 12.30 \text{ MN/m}^2$$

With half shear transfer ($\gamma = 0.5$):

$$\sigma_y = 5.72 + 3.50 = 9.22 \text{ MN/m}^2$$

With full shear transfer ($\gamma = 1$):

$$\sigma_y = 5.72 + 0.43 = 6.15 \text{ MN/m}^2$$

Comparison with Westergaard's results:

The deflection is given by equation (E-7) of [6]:

$$w = \frac{(1 + 0.40\mu) P}{\sqrt{6} kl^2} = 11.0 \cdot 10^{-4} \text{ m}$$

The maximum bending stress is given for a semi-circle with radius a_2 either by equation (E-1) from the ordinary theory (1926), either by equation (E-5) from the "new" theory (1948).

The radius of the semi-circle corresponding to a square load of 0.25 m^2 is $a_2 = 0.40 \text{ m}$.

Hence, using equation (E-1):

$$\begin{aligned}\sigma_y &= 0.529 (1+0.54\mu)P [\log_{10} [Eh^3/ka_2^4]/(h^2)] \\ &= 0.529 (1+0.54\mu)P \times \\ &\quad \times [\log_{10} [12(1-\mu^2)l^4/a_2^4]/(h^2)] \\ &= 9.72 \text{ MN/m}^2\end{aligned}$$

Using equation (E-5):

$$\begin{aligned}\sigma_y &= -6P(1+0.5\mu)[0.489\log_{10}(a_2/l) - \\ &\quad - 0.091 - 0.027(a_2/l)]/(h^2) \\ &= 12.23 \text{ MN/m}^2\end{aligned}$$

6.3. Bending stress function of the distance to the edge

We compute the value of the bending stress σ_y in function of the distance of the load to the edge in order to determine at which distance the influence of the edge can be neglected. We consider a joint without shear transfer.

The bending stress is given by (32) in which the distance $d - c$, is varied. The stress is computed for $x = c$.

One obtains:

$$\begin{aligned}\text{For } d - c = a = 0.25 \text{ m} \\ \sigma_y &= 6.73 + 3.88 = 10.61 \text{ MN/m}^2\end{aligned}$$

$$\begin{aligned}\text{For } d - c = 0.50 \text{ m} \\ \sigma_y &= 6.73 + 1.71 = 7.44 \text{ MN/m}^2\end{aligned}$$

$$\text{For } d - c = 1.00 \text{ m}$$

$$\sigma_y = 6.73 + 0.44 = 7.17 \text{ MN/m}^2$$

$$\text{For } d - c = 1.50 \text{ m}$$

$$\sigma_y = 6.71 + 0.15 = 6.86 \text{ MN/m}^2$$

6.4. Edge stress in a slab submitted to a tandem load

Consider the tandem load represented at figure 3.

With the until now available tools, the tandem load had to be assimilated to a single load of $0.50 \times 0.50 \text{ m}$ located at the edge of the slab.

The bending stress at the edge for a single load has been computed in par. 6.2.:

$$\sigma_y = 5.72 + 6.58 = 12.20 \text{ MN/m}^2$$

However relation (32) allows for the computation of the stress induced by both loads.

Hence for load I:

$$\sigma = 1.65 + 1.64 = 3.29 \text{ MN/m}^2$$

and for load II:

$$\sigma = 3.37 + 4.08 = 7.45 \text{ MN/m}^2$$

For both loads

$$\begin{aligned}\sigma &= 3.29 + 7.45 = 10.74 \text{ MN/m}^2 \\ &< 12.20 \text{ MN/m}^2\end{aligned}$$

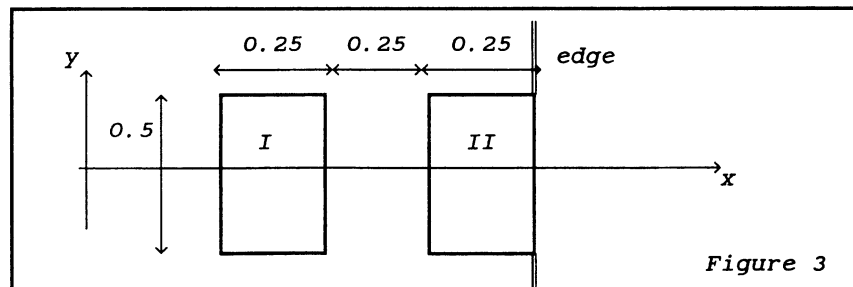


Figure 3

8. Conclusions

We have developed a general analytical solution for the problem of a concrete slab submitted to interior and edge loads with no, partial and full shear transfer at the edge.

The solution is based on mathematical models which accuracy has been checked, using finite element methods, for two-dimensionnal beams and circular slabs. The results confirm the validity of Westergaard's equations, as far as the 1948 formula for the edge load is applied (the 1926 formula underestimates the stress), and thus also confirm Ioannides et al [6] findings.

The advantage of the solution, besides that it has fully been demonstrated, lies in the fact that it allows for the computation of stresses and deflections at any distance from the load(s), whatever their location, with eventual shear transfer at the edge.

The computations are performed in no appreciable time using very simple computer programs.

The efficiency of the load transfer at the edge can be evaluated using Teller's and Sutherland's [7] first formula:

$$E = (\sigma_f - \sigma_j) / (\sigma_f - \sigma_i)$$

where σ_f is the stress at the free edge,

σ_j is the stress at the jointed edge,

σ_i is the stress at an interior point.

The efficiency E_σ can be estimated by the in situ measured deflection using a similar formula for the deflection efficiency

$$E = (w_f - w_j) / (w_f - w_i)$$

With the results of par. 6.2. one obtains:

With no shear transfer ($\gamma = 0$):

$$E_w = 0 \quad E_\sigma = 0$$

With half shear transfer ($\gamma = 0.5$):

$$E_w = 0.38 \quad E_\sigma = 0.45$$

With full shear transfer ($\gamma = 1$):

$$E_w = 0.75 \quad E_\sigma = 0.93$$

Hence, the determination of E_w allows for an estimate of E_σ .

Further (par. 6.3.), the solution allows to determine the required overwidth if one wants to avoid free-edge stresses.

Finally (par.6.4.), the solution allows for a more accurate determination of the edge stresses under tandem loads, which appear to be lower than usually estimated.

References

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Appendix 1

Partial resolution of the integral

$$W = \frac{4p}{\pi^2 k} \int_0^\infty \int_0^\infty \frac{\cos(tx/l) \sin(ta/l) \cos(sy/l) \sin(sb/l)}{ts(t^4 + 2t^2 s^2 + s^4 + 1)} ds dt$$

1.1. Resolution for $x > a$

The partial resolution of the double integral is based on a method proposed by Pronk [2], which consists in splitting the integral in two integrals with square exponents in the denominator, integrals which then can be solved by existing tools.

Hence we write

$$(t^2 + s^2)^2 + 1 = (t^2 + s^2 + i)(t^2 + s^2 - i) \\ = (t^2 + z_1^2)(t^2 + z_2^2)$$

Further we write

$$z_1 = \sqrt{s^2 + i} = \alpha + i\beta \\ z_2 = \sqrt{s^2 - i} = \alpha - i\beta$$

with

$$\alpha = \left[\frac{[s^4 + 1]^{1/2} + s^2}{2} \right]^{1/2} \\ \beta = \left[\frac{[s^4 + 1]^{1/2} - s^2}{2} \right]^{1/2}$$

The denominator is splitted

$$\frac{1}{t^4 + 2t^2 s^2 + s^4 + 1} = \frac{i}{2} \left[\frac{1}{t^2 + z_1^2} - \frac{1}{t^2 + z_2^2} \right]$$

and

$$W = \frac{4p}{\pi^2 k} \frac{i}{2} \int_0^\infty \int_0^\infty \frac{\cos(sy/l) \sin(sb/l)}{s} \times \left[\frac{\cos(tx/l) \sin(ta/l)}{t(t^2 + z_1^2)} - \frac{\cos(tx/l) \sin(ta/l)}{t(t^2 + z_2^2)} \right] ds dt$$

When $x > a$, the solution of the integral is given by Watson [3]

$$\int_0^\infty \frac{\cos(tx/l) \sin(ta/l)}{t(t^2 + z^2)} dt = \\ = \frac{\pi}{2} \frac{(ax)^{1/2}}{zl} I_{1/2}(az/l) K_{-1/2}(xz/l)$$

Hence

$$\frac{4p}{\pi^2 k} \frac{i}{2} \int_0^\infty \left[\frac{\cos(tx/l) \sin(ta/l)}{t(t^2 + z_1^2)} - \frac{\cos(tx/l) \sin(ta/l)}{t(t^2 + z_2^2)} \right] dt = \\ = \frac{p}{\pi k} \frac{(ax)^{1/2}}{l} i \left[\frac{I_{1/2}(az_1/l) K_{-1/2}(xz_1/l)}{z_1} - \frac{I_{1/2}(az_2/l) K_{-1/2}(xz_2/l)}{z_2} \right] \\ = \frac{p}{\pi k} \frac{i}{2} \left[\frac{e^{-(x-a)z_1/l} - e^{-(x+a)z_1/l}}{z_1^2} - \frac{e^{-(x-a)z_2/l} - e^{-(x+a)z_2/l}}{z_2^2} \right]$$

Writing

$$e^{-(x+a)z_1/l} = e^{-(x+a)(\alpha+i\beta)/l} \\ = e^{-(x+a)\alpha/l} \left[\cos\frac{(x+a)\beta}{l} + i \sin\frac{(x+a)\beta}{l} \right]$$

$$e^{-(x+a)z_2/l} = e^{-(x+a)(\alpha-i\beta)/l} \\ = e^{-(x+a)\alpha/l} \left[\cos\frac{(x+a)\beta}{l} - i \sin\frac{(x+a)\beta}{l} \right]$$

and simplifying, the integral finally becomes

$$w = \frac{p}{\pi k} \int_0^\infty \frac{\cos(sy/l)\sin(sb/l)}{s(s^4+1)} \left\{ e^{-(x-a)\alpha/l} \times \right. \\ \times \left[\cos\frac{(x-a)\beta}{l} + s^2 \sin\frac{(x-a)\beta}{l} \right] - e^{-(x+a)\alpha/l} \times \\ \times \left. \left[\cos\frac{(x+a)\beta}{l} + s^2 \sin\frac{(x+a)\beta}{l} \right] \right\} ds$$

1.2. Resolution for $x < a$

When $x < a$, the splitted integral has no solution by lack of convergence.

Hence we write

$$\cos(tx/l)\sin(ta/l) = \\ = \sin[t(x+a)/l] - \sin(tx/l)\cos(ta/l)$$

We then follow the same procedure as in par. 1.1. utilizing next relations [Watson, (3)]:

$$\int_0^\infty \frac{\sin(ta/l)}{t(t^2+z^2)} dt = \\ = \frac{\pi}{2} \frac{a}{zl} I_{1/2}(az/2l) K_{1/2}(az/2l)$$

$$\int_0^\infty \frac{\sin(tx/l)\cos(ta/l)}{t(t^2+z^2)} dt = \\ = \frac{\pi}{2} \frac{(ax)^{1/2}}{zl} I_{1/2}(xz/l) K_{-1/2}(az/l)$$

so that finally

$$w = \frac{p}{\pi k} \int_0^\infty \frac{\cos(sy/l)\sin(sb/l)}{s(s^4+1)} \left\{ 2 - \right. \\ - e^{-(a-x)\alpha/l} \times \left[\cos\frac{(a-x)\beta}{l} + s^2 \sin\frac{(a-x)\beta}{l} \right] - \\ - e^{-(x+a)\alpha/l} \times \left. \left[\cos\frac{(x+a)\beta}{l} + s^2 \sin\frac{(x+a)\beta}{l} \right] \right\} ds$$

1.3. Resolution for $x = a$

For $x = a$, both solutions reduce to

$$w = \frac{p}{\pi k} \int_0^\infty \frac{\cos(sy/l)\sin(sb/l)}{s(s^4+1)} \left\{ 1 - \right. \\ - e^{-2a\alpha/l} \times \left. \left[\cos\frac{2a\beta}{l} + s^2 \sin\frac{2a\beta}{l} \right] \right\} ds$$

Appendix 2

Resolution of the integral

$$w = \frac{P}{\pi^2 k l^2} \int_0^\infty \int_0^\infty \frac{1}{ts(t^4 + 2t^2 s^2 + s^4 + 1)} ds dt$$

The integral is easily solved in using next transformation

$$\begin{aligned} t &= \rho \cos \vartheta \\ s &= \rho \sin \vartheta \\ ds dt &= \rho d\rho d\vartheta \end{aligned}$$

Hence

$$w = \frac{P}{\pi^2 k l^2} \int_0^{\pi/2} \int_0^\infty \frac{\rho d\rho d\vartheta}{\rho^4 + 1} = \frac{P}{2\pi k l^2} \int_0^\infty \frac{\rho d\rho}{\rho^4 + 1}$$

Now write $\rho^2 = \tan \chi$, $2\rho d\rho = d\chi / \cos^2 \chi$

Hence

$$w = \frac{P}{4\pi k l^2} \int_0^{\pi/2} d\chi = \frac{P}{8 k l^2}$$

Appendix 3

The supplementary solutions of the homogeneous equation

$$\nabla^2 \nabla^2 w + \frac{k}{D} w = 0$$

In the more simple cases of a two-dimensionnal beam and a circular slab the supplementary solutions required to express a finite boundary are, as shown by Van Cauwelaert [4], the real and imaginary parts of the complex conjugate functions $\cos(xv \pm i)$, for the beam, and $J_0(rv \pm i)$, for the circular slab. These functions can be expressed as infinite series of the variates x or r .

The obtained analytical results were checked by Buyse and Jacques [5] using a finite element program.

However in the case of a rectangular slab, where the solutions depend on both variables x and y , one must search for functions able to express a finite boundary, let us say for $x = c$, whatever the value of y . Thus series, where x and y are necessarily related, cannot be used. Therefore we look for a solution by the method of separation of the variables, $w = f(x).g(y)$, with

$$g(y) = \int_0^\infty \cos(sy/l) \sin(sb/l) ds$$

The homogeneous equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] + \frac{w}{l^4} = 0$$

transforms into

$$\begin{aligned} \frac{1}{l^4} \int_0^\infty \left[\frac{\partial^4 f(x/l)}{\partial x^4} - 2s^2 \frac{\partial^2 f(x/l)}{\partial x^2} + \right. \\ \left. + (s^4 + 1)f(x/l) \right] \cos(sy/l) \sin(sb/l) ds = 0 \end{aligned}$$

and reduces in a differential equation of x only

$$\frac{\partial^4 f(x/l)}{\partial x^4} - 2s^2 \frac{\partial^2 f(x/l)}{\partial x^2} + (s^4 + 1)f(x/l) = 0$$

Writing the characteristic equation

$$D^4 - 2s^2 D^2 + (s^4 + 1) = 0$$

one obtains

$$D_{1,2} = \pm (\alpha + i\beta) \quad D_{3,4} = \pm (\alpha - i\beta)$$

where α and β are given by (14).

Hence, using the complex relations developed in appendix 1

$$f(x/l) = e^{\alpha x/l} [A \cos(\beta x/l) + B \sin(\beta x/l)] + e^{-\alpha x/l} [C \cos(\beta x/l) + D \sin(\beta x/l)]$$

Those solutions will be used for values of x between 0 and d , the absciss of the boundary. Therefore we consider only the positive exponents. Then the complete supplementary solutions become

$$w_A = \int_0^\infty A(s) e^{\alpha x/l} \cos(\beta x/l) \cos(ys/l) \sin(bs/l) ds$$

$$w_B = \int_0^\infty B(s) e^{\alpha x/l} \sin(\beta x/l) \cos(ys/l) \sin(bs/l) ds$$

which we write, for reasons of homogeneity with the solution for the infinite slab, as follows

$$w_A = \frac{p}{\pi k} \int_0^\infty A(s) e^{\alpha x/l} \times$$

$$\times \cos(\beta x/l) \cos(ys/l) \sin(bs/l) ds$$

$$w_B = \frac{p}{\pi k} \int_0^\infty B(s) e^{\alpha x/l} \times$$

$$\times \sin(\beta x/l) \cos(ys/l) \sin(bs/l) ds$$

wherein $A(s)$ and $B(s)$ are functions of s .

Appendix 4

Determination of the derivatives

4.1. Derivatives for the main solution

 For $0 \leq x \leq a$

$$\begin{aligned} \frac{dw_1}{dx} = & - \frac{p}{\pi k l} \int_0^{\infty} \frac{\cos(sy/l) \sin(sb/l)}{s (s^4 + 1)} \left\{ e^{-\alpha(a-x)/l} \times \right. \\ & \times \left[(\alpha - \beta s^2) \cos \frac{x-a}{l} \beta - (\alpha s^2 + \beta) \sin \frac{x-a}{l} \beta \right] - e^{-\alpha(a+x)/l} \times \\ & \times \left[(\alpha - \beta s^2) \cos \frac{x+a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{x+a}{l} \beta \right] \left. \right\} ds \end{aligned} \quad (4.1)$$

$$\begin{aligned} \frac{d^2 w_1}{dx^2} = & \frac{p}{\pi k l^2} \int_0^{\infty} \frac{\cos(sy/l) \sin(sb/l)}{s} \left\{ e^{-\alpha(a-x)/l} \sin \frac{x-a}{l} \beta - \right. \\ & \left. - e^{-\alpha(a+x)/l} \sin \frac{x+a}{l} \beta \right\} ds \end{aligned} \quad (4.2)$$

$$\begin{aligned} \frac{d^3 w_1}{dx^3} = & \frac{p}{\pi k l^3} \int_0^{\infty} \frac{\cos(sy/l) \sin(sb/l)}{s} \left\{ e^{-\alpha(a-x)/l} \left[\alpha \sin \frac{x-a}{l} \beta + \right. \right. \\ & \left. \left. + \beta \cos \frac{x-a}{l} \beta \right] + e^{-\alpha(a+x)/l} \left[\alpha \sin \frac{x+a}{l} \beta - \beta \cos \frac{x+a}{l} \beta \right] \right\} ds \end{aligned} \quad (4.3)$$

$$\begin{aligned} \frac{d^2 w_1}{dy^2} = & - \frac{p}{\pi k l^2} \int_0^{\infty} \frac{s \cos(sy/l) \sin(sb/l)}{(s^4 + 1)} \left\{ 2 - e^{-\alpha(a-x)/l} \left[\cos \frac{x-a}{l} \beta - \right. \right. \\ & \left. \left. - s^2 \sin \frac{x-a}{l} \beta \right] - e^{-\alpha(a+x)/l} \left[\cos \frac{x+a}{l} \beta + s^2 \sin \frac{x+a}{l} \beta \right] \right\} ds \end{aligned} \quad (4.4)$$

$$\begin{aligned} \frac{d^3 w_1}{dx dy^2} = & \frac{p}{\pi k l^3} \int_0^{\infty} \frac{s \cos(sy/l) \sin(sb/l)}{(s^4 + 1)} \left\{ e^{-\alpha(a-x)/l} \times \right. \\ & \times \left[(\alpha - \beta s^2) \cos \frac{x-a}{l} \beta - (\alpha s^2 + \beta) \sin \frac{x-a}{l} \beta \right] - e^{-\alpha(a+x)/l} \times \\ & \times \left[(\alpha - \beta s^2) \cos \frac{x+a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{x+a}{l} \beta \right] \left. \right\} ds \end{aligned} \quad (4.5)$$

 For $x > a$

$$\begin{aligned} \frac{dw_1}{dx} = & - \frac{p}{\pi k l} \int_0^{\infty} \frac{\cos(sy/l) \sin(sb/l)}{s (s^4 + 1)} \left\{ e^{-\alpha(x-a)/l} \times \right. \\ & \times \left[(\alpha - \beta s^2) \cos \frac{x-a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{x-a}{l} \beta \right] - e^{-\alpha(a+x)/l} \times \\ & \times \left[(\alpha - \beta s^2) \cos \frac{x+a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{x+a}{l} \beta \right] \left. \right\} ds \end{aligned} \quad (4.6)$$

$$\frac{d^2 w_i}{dx^2} = \frac{p}{\pi k l^2} \int_0^\infty \frac{\cos(sy/l) \sin(sb/l)}{s} \left\{ e^{-\alpha(x-a)/l} \sin \frac{x-a}{l} \beta - e^{-\alpha(a+x)/l} \sin \frac{x+a}{l} \beta \right\} ds \quad (4.7)$$

$$\frac{d^3 w_i}{dx^3} = \frac{p}{\pi k l^3} \int_0^\infty \frac{\cos(sy/l) \sin(sb/l)}{s} \left\{ e^{-\alpha(x-a)/l} \left[\beta \cos \frac{x-a}{l} \beta - \alpha \sin \frac{x-a}{l} \beta \right] + e^{-\alpha(a+x)/l} \left[\alpha \sin \frac{x+a}{l} \beta - \beta \cos \frac{x+a}{l} \beta \right] \right\} ds \quad (4.8)$$

$$\frac{d^2 w_i}{dy^2} = - \frac{p}{\pi k l^2} \int_0^\infty \frac{s \cos(sy/l) \sin(sb/l)}{(s^4 + 1)} \left\{ e^{-\alpha(x-a)/l} \left[\cos \frac{x-a}{l} \beta + s^2 \sin \frac{x-a}{l} \beta \right] - e^{-\alpha(a+x)/l} \left[\cos \frac{x+a}{l} \beta + s^2 \sin \frac{x+a}{l} \beta \right] \right\} ds \quad (4.9)$$

$$\begin{aligned} \frac{d^3 w_i}{dx dy^2} = & \frac{p}{\pi k l^3} \int_0^\infty \frac{s \cos(sy/l) \sin(sb/l)}{(s^4 + 1)} \left\{ e^{-\alpha(x-a)/l} \times \right. \\ & \times \left[(\alpha - \beta s^2) \cos \frac{x-a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{x-a}{l} \beta \right] - e^{-\alpha(a+x)/l} \times \\ & \times \left[(\alpha - \beta s^2) \cos \frac{x+a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{x+a}{l} \beta \right] \left. \right\} ds \end{aligned} \quad (4.10)$$

4.2. Derivatives for the supplementary solutions

$$\begin{aligned} \frac{dw_A}{dx} = & \frac{p}{\pi k l} \int_0^\infty A(s) e^{\alpha x/l} [\alpha \cos(\beta x/l) - \beta \sin(\beta x/l)] \times \\ & \times \cos(sy/l) \sin(sb/l) ds \end{aligned} \quad (4.11)$$

$$\begin{aligned} \frac{d^2 w_A}{dx^2} = & \frac{p}{\pi k l^2} \int_0^\infty A(s) e^{\alpha x/l} [s^2 \cos(\beta x/l) - \sin(\beta x/l)] \times \\ & \times \cos(sy/l) \sin(sb/l) ds \end{aligned} \quad (4.12)$$

$$\begin{aligned} \frac{d^3 w_A}{dx^3} = & - \frac{p}{\pi k l^3} \int_0^\infty A(s) e^{\alpha x/l} [(\alpha + \beta s^2) \sin(\beta x/l) - \\ & - (\alpha s^2 - \beta) \cos(\beta x/l)] \cos(sy/l) \sin(sb/l) ds \end{aligned} \quad (4.13)$$

$$\begin{aligned} \frac{d^2 w_A}{dy^2} = & - \frac{p}{\pi k l^2} \int_0^\infty s^2 A(s) e^{\alpha x/l} \cos(\beta x/l) \cos(sy/l) \sin(sb/l) ds \end{aligned} \quad (4.14)$$

$$\frac{d^3 w_A}{dx dy^2} = - \frac{p}{\pi k l^3} \int_0^{\infty} s^2 A(s) e^{\alpha x/l} [\alpha \cos(\beta x/l) - \beta \sin(\beta x/l)] \times \\ \times \cos(sy/l) \sin(sb/l) ds \quad (4.15)$$

$$\frac{dw_B}{dx} = \frac{p}{\pi k l} \int_0^{\infty} B(s) e^{\alpha x/l} [\alpha \sin(\beta x/l) + \beta \cos(\beta x/l)] \times \\ \times \cos(sy/l) \sin(sb/l) ds \quad (4.16)$$

$$\frac{d^2 w_B}{dx^2} = \frac{p}{\pi k l^2} \int_0^{\infty} B(s) e^{\alpha x/l} [s^2 \sin(\beta x/l) + \cos(\beta x/l)] \times \\ \times \cos(sy/l) \sin(sb/l) ds \quad (4.17)$$

$$\frac{d^3 w_B}{dx^3} = \frac{p}{\pi k l^3} \int_0^{\infty} B(s) e^{\alpha x/l} [(\alpha + \beta s^2) \cos(\beta x/l) + \\ + (\alpha s^2 - \beta) \sin(\beta x/l)] \cos(sy/l) \sin(sb/l) ds \quad (4.18)$$

$$\frac{d^2 w_B}{dy^2} = - \frac{p}{\pi k l^2} \int_0^{\infty} s^2 B(s) e^{\alpha x/l} \sin(\beta x/l) \cos(sy/l) \sin(sb/l) ds \\ (4.19)$$

$$\frac{d^3 w_B}{dx dy^2} = - \frac{p}{\pi k l^3} \int_0^{\infty} s^2 B(s) e^{\alpha x/l} [\alpha \sin(\beta x/l) + \beta \cos(\beta x/l)] \times \\ \times \cos(sy/l) \sin(sb/l) ds \quad (4.20)$$

Appendix 5Determination of the functions A(s) and B(s)

Consider the slab represented at figure 2. The boundary conditions are expressed for $x = d$; the axis coordinates of the load are $(c, 0)$.

$$\frac{d^2 w_1}{dx^2} = \frac{p}{\pi k l^2} \int_0^\infty \frac{\cos(sy/l) \sin(sb/l)}{s} \left\{ e^{-\alpha(d-c-a)/l} \sin \frac{d-c-a}{l} \beta - e^{-\alpha(d-c+a)/l} \sin \frac{d-c+a}{l} \beta \right\} ds$$

$$\frac{d^3 w_1}{dx^3} = \frac{p}{\pi k l^3} \int_0^\infty \frac{\cos(sy/l) \sin(sb/l)}{s} \left\{ e^{-\alpha(d-c-a)/l} \left[\beta \cos \frac{d-c-a}{l} \beta - \alpha \sin \frac{d-c-a}{l} \beta \right] + e^{-\alpha(d-c+a)/l} \left[\alpha \sin \frac{d-c+a}{l} \beta - \beta \cos \frac{d-c+a}{l} \beta \right] \right\} ds$$

$$\frac{d^2 w_i}{dy^2} = - \frac{p}{\pi k l^2} \int_0^\infty \frac{s \cos(sy/l) \sin(sb/l)}{(s^4 + 1)} \left\{ e^{-\alpha(d-c-a)/l} \left[\cos \frac{d-c-a}{l} \beta + s^2 \sin \frac{d-c-a}{l} \beta \right] - e^{-\alpha(d-c+a)/l} \left[\cos \frac{d-c+a}{l} \beta + s^2 \sin \frac{d-c+a}{l} \beta \right] \right\} ds$$

$$\begin{aligned} \frac{d^3 w_i}{dx dy^2} &= \frac{p}{\pi k l^3} \int_0^\infty \frac{s \cos(sy/l) \sin(sb/l)}{(s^4 + 1)} \left\{ e^{-\alpha(d-c-a)/l} \times \right. \\ &\times \left[(\alpha - \beta s^2) \cos \frac{d-c-a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{d-c-a}{l} \beta \right] - e^{-\alpha(d-c+a)/l} \times \\ &\times \left[(\alpha - \beta s^2) \cos \frac{d-c+a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{d-c+a}{l} \beta \right] \left. \right\} ds \end{aligned}$$

$$\begin{aligned} \frac{d^2 w_A}{dx^2} &= \frac{p}{\pi k l^2} \int_0^\infty A(s) e^{\alpha d/l} [s^2 \cos(\beta d/l) - \sin(\beta d/l)] \times \\ &\times \cos(sy/l) \sin(sb/l) ds \end{aligned}$$

$$\begin{aligned} \frac{d^3 w_A}{dx^3} &= - \frac{p}{\pi k l^3} \int_0^\infty A(s) e^{\alpha d/l} [(\alpha + \beta s^2) \sin(\beta d/l) + \\ &+ (\alpha s^2 - \beta) \cos(\beta d/l)] \cos(sy/l) \sin(sb/l) ds \end{aligned}$$

$$\frac{d^2 w_A}{dy^2} = - \frac{p}{\pi k l^2} \int_0^\infty s^2 A(s) e^{\alpha d/l} \cos(\beta d/l) \cos(sy/l) \sin(sb/l) ds$$

$$\begin{aligned} \frac{d^3 w_A}{dx dy^2} &= - \frac{p}{\pi k l^3} \int_0^\infty s^2 A(s) e^{\alpha d/l} [\alpha \cos(\beta d/l) - \beta \sin(\beta d/l)] \times \\ &\times \cos(sy/l) \sin(sb/l) ds \end{aligned}$$

$$\begin{aligned}
 \frac{d^2 w_B}{dx^2} &= \frac{p}{\pi k l^2} \int_0^\infty B(s) e^{\alpha d/l} [s^2 \sin(\beta d/l) + \cos(\beta d/l)] \times \\
 &\quad \times \cos(sy/l) \sin(sb/l) ds \\
 \frac{d^3 w_B}{dx^3} &= \frac{p}{\pi k l^3} \int_0^\infty B(s) e^{\alpha d/l} [(\alpha + \beta s^2) \cos(\beta d/l) + \\
 &\quad + (\alpha s^2 - \beta) \sin(\beta d/l)] \cos(sy/l) \sin(sb/l) ds \\
 \frac{d^2 w_B}{dy^2} &= - \frac{p}{\pi k l^2} \int_0^\infty s^2 B(s) e^{\alpha d/l} \sin(\beta d/l) \cos(sy/l) \sin(sb/l) ds \\
 \frac{d^3 w_B}{dx dy^2} &= - \frac{p}{\pi k l^3} \int_0^\infty s^2 B(s) e^{\alpha d/l} [\alpha \sin(\beta d/l) + \beta \sin(\beta d/l)] \times \\
 &\quad \times \cos(sy/l) \sin(sb/l) ds
 \end{aligned}$$

The boundary conditions must be verified for any value of y .
Hence (25) is verified if

$$\begin{aligned}
 &- \frac{1}{s} \left[e^{-(d-c+a)\alpha/l} \sin \frac{d-c+a}{l} \beta - e^{-(d-c-a)\alpha/l} \sin \frac{d-c-a}{l} \beta \right] \\
 &+ \frac{s\mu}{(s^4 + 1)} \left\{ e^{-(d-c-a)\alpha/l} \left[\cos \frac{d-c-a}{l} \beta + s^2 \sin \frac{d-c-a}{l} \beta \right] - \right. \\
 &\left. - e^{-(d-c+a)\alpha/l} \left[\cos \frac{d-c+a}{l} \beta + s^2 \sin \frac{d-c+a}{l} \beta \right] \right\} \\
 &- \mu s^2 A(s) e^{\alpha d/l} \cos(\beta d/l) + B(s) e^{\alpha d/l} [s^2 \sin(\beta d/l) + \cos(\beta d/l)] \\
 &- \mu s^2 B(s) e^{\alpha d/l} \sin(\beta d/l) = 0
 \end{aligned}$$

Relation (27) is verified if

$$\begin{aligned}
 &(1-\gamma) \left\{ \frac{1}{s} e^{-(d-c-a)\alpha/l} \left[\beta \cos \frac{d-c-a}{l} \beta - \alpha \sin \frac{d-c-a}{l} \beta \right] - \right. \\
 &- \frac{1}{s} e^{-(d-c+a)\alpha/l} \left[\beta \cos \frac{d-c+a}{l} \beta - \alpha \sin \frac{d-c+a}{l} \beta \right] + \frac{s(2-\mu)}{(s^4 + 1)} \times \\
 &\times \left[e^{-(d-c-a)\alpha/l} \left[(\alpha - \beta s^2) \cos \frac{d-c-a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{d-c-a}{l} \beta \right] - \right. \\
 &- e^{-(d-c+a)\alpha/l} \left[(\alpha - \beta s^2) \cos \frac{d-c+a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{d-c+a}{l} \beta \right] \left. \right] - \\
 &- A(s) e^{\alpha d/l} [(\alpha + \beta s^2) \sin(\beta d/l) + (\alpha s^2 - \beta) \cos(\beta d/l)] - \\
 &- (2 - \mu) s^2 A(s) e^{\alpha d/l} [\alpha \cos(\beta d/l) - \beta \sin(\beta d/l)] +
 \end{aligned}$$

$$+ B(s) e^{\alpha d/l} [(\alpha + \beta s^2) \cos(\beta d/l) + (\alpha s^2 - \beta) \sin(\beta d/l)] - \\ - (2 - \mu) s^2 B(s) e^{\alpha d/l} [\alpha \sin(\beta d/l) + \beta \sin(\beta d/l)] = 0$$

The system is reduced to

$$A(s) e^{\alpha d/l} [s^2 (1 - \mu) \cos(\beta d/l) - \sin(\beta d/l)] + \\ + B(s) e^{\alpha d/l} [s^2 (1 - \mu) \sin(\beta d/l) + \cos(\beta d/l)] = F_1 \\ - A(s) e^{\alpha d/l} [(1 - \mu) \alpha s^2 + \beta] \cos(\beta d/l) - [(1 - \mu) \beta s^2 - \alpha] \sin(\beta d/l) - \\ - B(s) e^{\alpha d/l} [(1 - \mu) \beta s^2 - \alpha] \cos(\beta d/l) + [(1 - \mu) \alpha s^2 + \beta] \sin(\beta d/l) = F_2$$

where

$$F_1 = \frac{1}{s} \left[e^{-(d-c+a)\alpha/l} \sin \frac{d-c+a}{l} \beta - e^{-(d-c-a)\alpha/l} \sin \frac{d-c-a}{l} \beta \right] \\ + \frac{s\mu}{(s^4 + 1)} \left\{ e^{-(d-c-a)\alpha/l} \left[\cos \frac{d-c-a}{l} \beta + s^2 \sin \frac{d-c-a}{l} \beta \right] - \right. \\ \left. - e^{-(d-c+a)\alpha/l} \left[\cos \frac{d-c+a}{l} \beta + s^2 \sin \frac{d-c+a}{l} \beta \right] \right\} \\ F_2 = - (1 - \gamma) \left\{ \frac{1}{s} e^{-(d-c-a)\alpha/l} \left[\beta \cos \frac{d-c-a}{l} \beta - \alpha \sin \frac{d-c-a}{l} \beta \right] - \right. \\ - \frac{1}{s} e^{-(d-c+a)\alpha/l} \left[\beta \cos \frac{d-c+a}{l} \beta - \alpha \sin \frac{d-c+a}{l} \beta \right] + \frac{s(2 - \mu)}{(s^4 + 1)} \times \\ \times \left[e^{-(d-c-a)\alpha/l} \left[(\alpha - \beta s^2) \cos \frac{d-c-a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{d-c-a}{l} \beta \right] - \right. \\ \left. - e^{-(d-c+a)\alpha/l} \left[(\alpha - \beta s^2) \cos \frac{d-c+a}{l} \beta + (\alpha s^2 + \beta) \sin \frac{d-c+a}{l} \beta \right] \right] \right\}$$

The resolution is given by

$$A(s) e^{\alpha d/l} = - \frac{[F_3 \cos(\beta d/l) + F_4 \sin(\beta d/l)]}{\Delta}$$

$$B(s) e^{\alpha d/l} = \frac{[F_4 \cos(\beta d/l) - F_3 \sin(\beta d/l)]}{\Delta}$$

$$\Delta = 2\alpha s^2 (1 - \mu) + \beta [1 - (1 - \mu)^2 s^4]$$

$$F_3 = [F_1 [(1 - \mu) \beta s^2 - \alpha] + F_2]$$

$$F_4 = [F_1 [(1 - \mu) \alpha s^2 + \beta] + F_2 (1 - \mu) s^2]$$