GLASNOST AND CONCRETE PAVEMENT ENGINEERING: DEVELOPMENTS CONCERNING SLABS-ON-GRADE FROM THE FORMER SOVIET UNION

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Advances in the state-of-knowledge concerning the behavior of slabs-on-grade achieved in the former Soviet Union are presented and discussed. Most of these are at present solutions to applied mathematics problems, but may be rendered applicable to concrete pavement engineering if recast in more practical forms using the capabilities of high speed computers. The Paper focusses on issues pertaining to the study of edge loading and load transfer phenomena in multi-layered concrete pavement systems. Contributions from the East are currently considered for incorporation in a mechanistic design procedure against catastrophic modes of failure ("limit states"), operative in addition to long-term, fatigue-type distress mechanisms.

INTRODUCTION

Following the recent historic events in Eastern Europe, a significant body of knowledge has become accessible to the West, resulting from research conducted over the last thirty years in Russia and other republics of the former Soviet Union into the behavior of slabs-on-grade. This development increases the potential for rapid technological progress in pavement engineering, particularly in the formulation of efficient design algorithms for Portland cement concrete (PCC) pavement systems. This Paper presents a review of several major advances in Soviet knowledge over the last thirty years, which may be rendered applicable to pavement analysis and design in the U.S. today.

American and Russian research activities in pavement engineering followed largely parallel paths until the early 1960s, when a significant divergence began to occur. This may be attributable to the technological revolution precipitated in the West by the introduction and routine use of the computer, and to our ability to afford extensive field tests, such as the AASHO Road Test and, more recently, the Strategic Highway Research Program (SHRP). Lacking such resources, and operating in a political system which did not provide the incentives of a free market economy, Soviet engineers were content to provide what were essentially solutions to applied mathematics problems. Whereas in the West the computer has been instrumental in most analysis and design developments, the East had been essentially forced to persist with the more traditional and tedious approach of mathematical closed-form derivations. Consequently, considerable short-term gains may now be achieved by using the computational capacity of the West in implementing some of the mathematical solutions from the East in the practice of pavement engineering.

The main thrust guiding this investigation is the observation that field performance data often suggest that whereas some pavements may crack at a relatively early age, others outlive their useful design periods exhibiting limited amounts of structural deterioration. Current design procedures based on the assumption that distresses accumulate slowly with repeated stress or strain applications cannot explain such failures (nor successes), suggesting that other modes of behavior may also need to be examined. Such reasoning leads to the adoption of a "Limit State" concept for the design of pavements, not unlike the one used "to ensure the structural integrity of aircraft designed for flight through turbulent air" *(1)*. Furthermore, theoretical considerations *(2)* and experimental results *(3)* indicate that edge loading conditions are critical and should be considered in concrete pavement design.

It is important that in the development of a mechanistic Limit State Pavement Design (LSPD) for PCC pavements, a number of pervasive issues associated with the conventional approach to solving edge loading problems should be addressed. These include: reproduction and verification of the conventional plate-on-dense liquid solution *(2)*; adoption of a subgrade model dispensing with the weaknesses of both the dense liquid (DL) and the elastic solid (ES) idealizations; analysis of multi-layered concrete pavement systems in a manner analogous to that used for asphalt concrete (AC) pavements *(4)*; development of efficient algorithms accounting for load transfer; and, definition of "limit states," operative in addition to long-term, fatigue-type distress mechanisms, and formulation of design defenses against them. Space limitations permit only the discussion of a few selected contributions from the former Soviet Union pertinent to these objectives. A much larger volume of information has been accumulated in this

study. Good reviews in English of Russian works in this field may also be found elsewhere *(5; 6; 7)*.

TOWARD AN IMPROVED SUBGRADE SOIL MODEL

The most widely adopted mechanistic idealization for the edge loading condition in concrete pavements is that of a plate on a DL foundation *(2)*, usually implemented in a finite element program, such as *ILLI-SLAB (8)*, *WESLIQID (9)*, *J-SLAB (10)*, and *FEACONS III (11)*. The advantage of this approach is that it allows consideration of the critical load transfer phenomena, occurring at the PCC slab joints, and the concomitant development of major distress types, such as faulting, pumping and corner breaking. The ES subgrade idealization, often considered to be a more realistic representation of real soils, is unsuitable for modeling the discontinuity of the deflection profile arising at the joints.

The discussion of improved subgrade models begins with the realization that neither the DL nor the ES idealizations are entirely adequate when applied to real soils, and that predictions from each exhibit discrepancies when compared to observed in situ behavior. The DL model assumes no shear interaction at all between adjacent spring elements and results in a foundation parameter, k, which is sensitive to the size of the plate used in its determination. On the other hand, the ES model ascribes to the foundation a higher degree of shear interaction than is usually available in the field, and results in infinite stress predictions under the edges and corners of a plate resting on it. Furthermore, real soil deflections vanish more rapidly than predicted by the ES, but, of course, not nearly as fast as assumed in the DL formulation.

Just as in the West, the plate load test (PLT) is recognized in Soviet literature as the primary source of information concerning the numerical values to be assigned to the constants describing each foundation model. Nonetheless, a long series of additional foundation models have been employed in the East, where the shortcomings of both the DL and ES are particularly emphasized. A concise description of eight major subgrade idealizations from a structural viewpoint, as well as a careful study of extensive PLT results, is presented by Cherkasov *(12)*. That investigator asserts that the mechanical properties of soil are extremely complex and defy exact descriptions. For design purposes, therefore, real soil foundations must be idealized using simple, approximate mechanical models. On the basis of numerous field loading tests, Cherkasov concludes that interpretation of PLT data on the basis of the maximum deflection alone is inadequate, and that the formulation of reliable subgrade models necessitates consideration of the entire surface deflection basin and of the soil response with depth. Table 1 is reproduced from Reference *(12)*; brief discussions of the models considered are presented below.

The DL or Winkler foundation *(13)* is the simplest foundation model and requires only one parameter, the subgrade modulus, k, which is the proportionality constant between the applied pressure and the load plate deflection. Subgrade deformations are local in character, i.e., they develop only beneath the load plate. Furthermore, they are elastic or recoverable upon load removal. The model ascribed to Fuss *(14)* is very similar to Winkler's, except for the fact that subgrade deformations are assumed to be non-recoverable.

In contrast, deformations in the ES Boussinesq half-space model are global in character, i.e., they develop not only under the load plate but also beyond it. All deformations are linearly elastic. In treating slab-on-grade problems, this model can also be considered as uni-parametric since the only parameter required is coefficient C, defined as:

$$C = \frac{E_s}{1 - \mu_s^2} \qquad (1)$$

in which E_S is the elastic modulus of the subgrade, and μ_S its Poisson's ratio. The Linear Half-Space model differs from the Elastic Half-Space in treating deformations occurring both under the load plate and beyond it as partially non-recoverable.

Deformations in the Finite Depth Layer model can be elastic or merely linear, although only the former approach is mentioned by Cherkasov *(12)*. Thus, this model is similar to the two Half-Space idealizations discussed, except inasmuch as deformations also depend on the finite layer thickness, H_F. Cherkasov *(12)* considers this model to be the most suitable for pavement subgrades, where significant deformations occur only in the upper soil layer rendered more compressible through cycles of wetting and drying, freezing and thawing. It should be noted, however, that like the half-space models, the finite layer model predicts infinite stresses under the edges and corners of a slab resting on it.

The two-parameter model by Pasternak *(15)* involves a compression coefficient, k_1, and a shear coefficient, G. Setting G to zero reduces this model to the DL foundation. Pasternak's model is also similar to the ES and Finite Layer models inasmuch as global deformations are predicted. Deflections beyond the load plate in a Pasternak subgrade vanish much faster than predicted by the ES model. Nonetheless, concentrated forces of an even greater magnitude than for the ES foundation are predicted under free, unloaded edges of slabs resting on a Pasternak foundation.

The model proposed by Zhemochkin and Sinitsyn *(16)* is another two-parameter idealization, and results from an in-series combination of the DL and ES models. It consists of a bed of independent springs of (areal) stiffness, k, resting on an elastic half-space characterized by the parameter C, defined by Equation 1. The spring deformations are considered non-recoverable and simulate the plastic component of real soil behavior, whereas the elastic half-space serves to model the resilient part of soil deflections. Such an idealization may be inefficient, however, since plastic deformations usually occur only along the slab edges and not under the central portion of the slab. A simpler model may be obtained by ignoring the plastic deformations and treating the deflections of both components as elastic, as suggested by the Russian Shtaerman *(17)*, as well as by Hetenyi *(18)* in the West. This simpler linear model, abbreviated in this Paper as ZSS, dispenses with the need to use plasticity concepts when modeling deflection profile discontinuities, such

as at joints between loaded and unloaded slabs, or when simulating a fracture within the soil mass.

The Cherkasov-Klein model *(19; 20)* is an example of a more complex idealization requiring three parameters: hardness number, A; hardness exponent, n; and elastic coefficient, C, as defined by Equation 1. A nonlinear relationship is assumed between the applied pressure and the deflection beneath the load plate, whereas deflections beyond the load plate are treated as linear.

The choice of subgrade model should be based on the following information extracted from PLT results: thickness of compressible layer (if finite) and variation of compressibility with depth; deflection basin beyond the plate edges; and variation with applied load of elastic and plastic deformation components occurring under the plate. As the thickness of the compression layer decreases, soil behavior tends to that predicted by the Winkler and Fuss models. It is interesting to note that Cherkasov *(12)* does not consider the ES model to be a desirable idealization, primarily because deflections beyond the load plate edges do not vanish as rapidly as observed in the field.

The eight models considered by Cherkasov illustrate the conventional approach to model formulation, according to which each model has a mechanical or physical interpretation, and is derived by some combination of ES and DL elements. Thus, the ZSS model consists of a connection in series of the ES and DL foundations. A model resulting from a connection in parallel of the two extreme idealizations has been proposed by Repnikov *(21; 22)*, and involves the ES reinforced by distributed frictionless springs. Although such a model would simulate surface deflections more realistically than the ES alone, it still predicts infinite stresses under the free edges of a slab resting on it *(23)*.

Korenev *(24)* suggests that a "mathematical" interpretation of these mechanical models is also possible, through the introduction of an influence function describing the main foundation characteristics in terms of its Fourier or Hankel transforms. A similar mathematical interpretation of the DL and ES models was presented in the West by Losberg *(25)*. A foundation influence function or kernel is simply the surface displacement profile w(x,y) resulting from the application of a unit concentrated load at a point (x_1,y_1). For any linear and isotropic foundation, this kernel depends only on the distance, R, between the load and the point considered. The power of a mathematical interpretation, however, hinges on its ability to formulate new models, which may not have any mechanical equivalents, in a manner characteristic of Russian theoreticians, and which may address effectively the development of infinite edge stresses. Table 2 presents the influence functions and their Fourier transforms for a variety of mechanical models, as well as for three mathematical models developed by Korenev *(24)*, Petrosyan *(26)*, and Klein *(27)*.

The ZSS model deserves a more detailed discussion because it appears to be the preferred choice not only of Cherkasov (12), but also of Gorbunov-Possadov (28) and Korenev (29). This model is related to both the ES and DL subgrades since it allows a limited degree of interaction between adjacent foundation elements, yet prevents the generation of infinite edge stresses, while at the same time remaining reasonably simple to formulate. In a conceptual sense, the ZSS model is the reverse of the better known two-parameter models, which add shear interactions to a discontinuous medium, since it effectively reduces the shear interaction provided by the entirely continuous ES subgrade.

The following expression has been obtained in this study for the surface deflection profile of a slab resting on a ZSS foundation loaded by an interior load, P, applied over a circular area radius a:

$$w(r) = \frac{P}{\pi a}\int_0^\infty \frac{\left(\frac{1}{k}+\frac{2}{C\alpha}\right)J_1(\alpha a)\,J_0(\alpha r)}{1+\frac{2D}{C}\alpha^3+\frac{D}{k}\alpha^4}\,d\alpha \qquad (2)$$

where D is the plate flexural stiffness defined as:

$$D = \frac{E\,h^3}{12\,(1-\mu^2)} \qquad (3)$$

Here h, E and μ denote the thickness, modulus of elasticity and Poisson's ratio of the slab, respectively.

The ZSS model is often found to reproduce in-situ soil behavior better than either the DL or the ES subgrades; it can be reduced to the DL or the ES foundations by the appropriate choice of model parameters; and it can also be incorporated in a finite element program such as *ILLI-SLAB (30)*. Like the DL subgrade, the ZSS model permits a deflection profile discontinuity at a loaded slab edge due to load transfer, but unlike the ES foundation it predicts no infinite stresses at a free, unloaded slab edge. The only other approach that would allow this would be a fully three dimensional finite element program incorporating special interface elements formulated on the basis of plasticity theory. It is anticipated that such a program will be the product of an ambitious, long-term initiative for the development of a Unified Pavement Design and Analysis Procedure (UPDAP), currently pursued by the Federal Aviation Administration (FAA). Because of the scope and lengthy duration of this FAA project, however, less demanding approaches, such as the slab-on-ZSS model examined in this investigation, continue to be very attractive.

MODELS FOR THE CONSTRUCTED LAYERS

A major weakness of the conventional approach to the solution of edge loading problems is the inability of current finite element programs to accommodate multi-layered PCC pavement systems in a manner analogous to that followed for AC pavements. One or two constructed pavement layers may be modeled (e.g., PCC slab and base, or PCC slab and AC overlay), but plate theory is used for both, thus ignoring compression through their thicknesses. Issues related to modeling multi-layered PCC pavements are considered in this section.

Pavements are usually constructed using local, low cost materials whose properties are highly variable and dependent on time, loading history and

environmental conditions *(31)*. Ideally, the behavior of these materials should be described using nonlinear, viscoplasticity concepts accounting for numerous random discontinuities. Such an approach is at present quite unfeasible; in fact, the complexity of the pavement problem and the uncertainties involved in its definition do not justify the use of any but the simplest idealizations conceivable. Over the last several decades, the collective experience of a great number of well-known investigators has pointed to the fact that constructed layers in a pavement system can be modeled adequately in many cases as linear elastic materials.

Three different models are used most commonly in idealizing the constructed layers in a pavement system. These are: the Kirchhoff medium-thick plate *(32)*; the Reissner thick plate *(33)*; and the more general Burmister homogeneous and isotropic layer *(4)*. The main assumptions of the Kirchhoff plate model are as follows:

(a) Lines normal to the middle surface before deformation remain straight, unstretched and normal to the middle surface of the deformed plate;
(b) There is no deformation in the mid-plane of the plate, i.e., this plane remains neutral during bending; and
(c) The normal stress in the transverse direction can be disregarded.

In contrast, Reissner plate theory rejects assumptions (a) and (c) and introduces instead the assumption of a linear law for the through-the-thickness distribution of the stress components σ_x, σ_y and τ_{xy}. The governing differential equation for the Reissner plate on DL subgrade is similar to the corresponding equation describing the behavior of a Kirchhoff plate on a two-parameter (Pasternak) foundation. The respective solutions, however, are different due to the difference in boundary conditions. Finally, the Burmister elastic layer model imposes none of the plate theory assumptions. Stress and displacement components are written in terms of the biharmonic Airy stress function, Φ, which is evaluated by satisfying the boundary conditions.

Glazyrin *(34; 35)* considered the problem of a concrete slab resting on a DL or an ES foundation, and compared solutions obtained with each of the aforementioned three common slab models. Citing Korenev *(7)*, Glazyrin employed available interior loading deflection formulae for the Kirchhoff cases, known in the West after Westergaard *(36)* and Losberg *(25)*, and compared these to his own solutions pertaining to the Reissner plate and Burmister layer. It should be noted here that the Burmister-on-DL solution was recently obtained independently in the West by Van Cauwelaert *(37)*. Glazyrin concluded that Kirchhoff plate theory gives adequately accurate results in most cases. In fact, when the radius of the applied load is smaller than the thickness of the slab, Kirchhoff and Reissner theories produce almost the same error when compared to the "exact" Burmister layer solution, although Reissner overestimates the deflection whereas Kirchhoff underestimates it. As a result, Reissner's plate possesses no significant advantages over Kirchhoff's simpler model, and need not be used for practical purposes in the analysis of slabs-on-grade.

In this study, comparison of bending stresses obtained using Westergaard's "special theory" *(38)* to those resulting from the Burmister layer-on-DL idealization revealed better agreement than is obtained using "ordinary theory," i.e., with Kirchhoff's plate. This finding can be considered as a justification of Westergaard's "special theory" according to which the radius of the applied load, a, is replaced by a fictitious radius b, given by:

$$b=\sqrt{1.6a^2+h^2}-0.675h \qquad (4)$$

where h is the slab thickness. In a previous paper *(39)*, preference was expressed for the "ordinary" theory, but that should be interpreted as merely confirming that only "ordinary" theory reproduces the solution for a Kirchhoff plate supported by a DL foundation. The solution obtained in this study based on the Burmister layer, verifies that "special" theory was probably an effort on the part of Westergaard to account for the departure of concrete slabs from Kirchhoff plate behavior in some instances.

Another interesting formulation is presented by Totsky *(40)* who suggests treating a multi-layered system overlying a DL foundation as a series of alternating plate elements and springs. The plate elements model the bending, while the springs accommodate the direct compression occurring in such a system. Totsky states that for many concrete pavements, a transformation which results in an equivalent thickness reflecting the contributions of both the PCC slab and the base may not be valid, since the lower layer may be cracked or become debonded, leading to the generation of interface voids upon bending. His proposal is to insert a series of springs between the pavement layers, with spring stiffnesses defined as follows:

$$k_0 = K\left(\frac{E_i\, E_{i+1}}{h_i\, E_{i+1} + h_{i+1}\, E_i}\right) \qquad (5)$$

in which K = 2.112 to 2.461 (the latter being Totsky's own recommendation), and subscripts i and i+1 denote the layer just above and just below the springs, respectively. Totsky *(40)* then considers an example concerning a two-layered beam in which the lower layer is cracked, and demonstrates that the upper beam experiences a considerable increase in bending stress. In conclusion, he states that PCC overlays with mismatched joints should be avoided.

Totsky's model is well-suited for a study of layer interface void effects and of reflective cracking. Thus, it may be applied to the study of a multi-layered system in which consecutive layers experience debonding, as for example due to a temperature variation through the thickness, or due to cracking of an underlying layer. In this study, a three-layer system was considered, consisting of two constructed layers resting on a DL foundation of stiffness, k. Following Totsky, this system may be modeled by two plates, separated by a series of fictitious springs with stiffness, k_0, resting on the same DL of stiffness, k. The interface reaction pressure offered by the fictitious springs, q_0, is assumed to be proportional to the

difference of the deflections of the neutral axes of the upper and lower plates. Thus, two simultaneous differential equations for the two plates can be written, expressing the fact that q_0 is a reaction pressure under the upper layer, as well as an applied pressure acting at the top of the lower layer. Solution of these equations yields the two plate deflection distributions.

In this study, a modified approach was also developed for calculating the fictitious spring stiffness, denoted by k_0^M. Accordingly, Equation 2 is replaced by:

$$k_0^M = \frac{1}{\frac{1}{k_{01}} + \frac{1}{k_{02}}} \quad (6)$$

in which:

$$k_{01} = \frac{2E_1(1-\mu_1)}{h_1(1-\mu_1-2\mu_1^2)} \quad (7)$$

and

$$k_{02} = \frac{2E_2(1-\mu_2)}{h_2(1-\mu_2-2\mu_2^2)} \quad (8)$$

Subscripts 1 and 2 denote here the structural characteristics of the top and bottom constructed layers, respectively.

The maximum bending stress predictions obtained at the bottom of layer 1 using the original and the modified Totsky models for interior loading were compared to a numerical solution for two unbonded Burmister layers in full contact with a DL foundation -developed in this study by adapting the procedures outlined by Glazyrin *(35)* and Van Cauwelaert *(37)*- and to the "closed-form" approach presented by Ioannides, et al. *(41)*, as shown in Table 3. For all cases considered, the modified Totsky solution is closer to the Burmister-on-DL solution than either the original Totsky or the "closed-form" approach. Incorporation of the modified Totsky method into a finite element analysis of debonded layers due to a temperature differential is under development at this time. The development of both a forward and a back-calculation procedure for a multi-layered PCC pavement system resting on a DL or ZSS foundation is also pursued.

EDGE LOADING CONSIDERATIONS

Although the edge loading condition is generally recognized as critical in the design of PCC pavements, a discrepancy between Westergaard's original equation for the edge stress *(42)* and his more recent formula *(2)* went unnoticed for several decades *(39)*, and remains unexplained to this day. In fact, the formulation leading to Westergaard's more rigorous 1948 formula has been impossible to reproduce [see, for example, Reference *(25)*]. Furthermore, conventional finite element analysis of edge loading requires considerable skill and computer resources, can be time consuming, and does not lend itself well for incorporation into a design guide. A very powerful approach to the solution of edge loading problems has been proposed by Soviet investigators and offers the potential of achieving significant breakthroughs in addressing these shortcomings. Called the "Method of Compensative Loads" (MCL), it is presented below following closely the book by Korenev and Chernigovskaya *(43)*. A brief general description of MCL is provided in Korenev's earlier book *(24)*.

To begin with, consider a semi-infinite plate resting on a DL foundation and loaded by a concentrated load, P, acting at an offset, $\bar{D}$, from the edge. This load may be considered to be an edge load if offset, $\bar{D}$, does not exceed 2ℓ, where ℓ is the radius of relative stiffness of the plate-subgrade system. The solution of this problem may be obtained as the sum of two individual components:

(a) The *primary solution*, pertaining to an infinite plate of identical structural properties and loaded by the same load as the original semi-infinite plate. The load is treated as acting at the interior of the infinite plate. This solution satisfies the equilibrium equation of the original system, as well as its boundary conditions at infinity. Naturally, it does not satisfy the free edge boundary conditions, which specify that vertical shearing forces and bending moments along the free edge should vanish;
(b) The *compensative solution*, which is added in order to rectify the violation of the free-edge boundary conditions by the primary solution. The compensative solution should also satisfy the governing differential equation for the original plate-subgrade system.

Korenev suggests replacing the applied concentrated edge load by a line of concentrated loads acting parallel to the free edge, also at an offset $\bar{D}$. Each of these individual loads is of the same magnitude, P, as the original load. The spacing between the individual loads is set to 2b, a value that is large enough so that the response under each load is not influenced significantly by the presence of the other loads. The numerical value of 2b used by Korenev is 8ℓ. For the case of the DL foundation, which provides for no shear interaction between adjacent subgrade elements, Korenev suggests also considering a second semi-infinite plate that is the mirror image of the original plate subjected to another line of concentrated loads, reflected along the free edge, with no load transfer along the joint thus formed. The solution to the resulting system of two independent semi-infinite plates under two lines of concentrated loads may then be assumed to be the same as the solution to the original problem of one semi-infinite plate under a single concentrated load. This is a consequence of the fact that for the case of the DL foundation, the additional line of loads introduced on the second semi-infinite plate does not alter the response of the original semi-infinite plate. The primary solution is the solution obtained when the free edge is ignored, and a single infinite plate is considered, subjected to two lines of loads at its interior. The purpose of the compensative solution is to eliminate the bending moments along the symmetry line, which corresponds to the free edge in the two-plate system.

Primary Solution

Consider first an infinite plate under one line of concentrated loads applied at its interior. This loading may be represented by the following Fourier series:

$$p(x) = \frac{P}{2b} + \frac{P}{b}\sum_{n=1}^{\infty} \cos\frac{n\pi x}{b} \tag{9}$$

in which x is measured parallel to the line of loads, with the origin under the original concentrated load. The solution for this line of loads may be obtained by summing the solutions for each elemental load term, $p_n(x)$, defined by:

$$p_n(x) = \frac{P}{b}\cos\frac{n\pi x}{b} \tag{10}$$

The deflection at any point (x,ψ) due to the nth term of the series may be written as:

$$w^p_n(x,\psi) = \bar{w}^p_n(\psi)\cos\frac{n\pi x}{b} \tag{11}$$

in which superscript p indicates that this deflection is part of the primary solution, and $\bar{w}^p_n(\psi)$ is an unknown function of ψ only to be determined. Coordinate ψ denotes the absolute value of the perpendicular distance of the point considered from the line of loads.

Substitution of Equation 11 into the differential equation for plate bending results in the following governing differential equation in terms of the unknown function $\bar{w}^p_n(\psi)$:

$$\frac{d^4\bar{w}^p_n}{d\psi^4} - 2\frac{n^2\pi^2}{b^2}\frac{d^2\bar{w}^p_n}{d\psi^2} + \left(\frac{n^4\pi^4}{b^4}+\frac{k}{D}\right)\bar{w}^p_n = 0 \tag{12}$$

in which k is the subgrade modulus and D is the plate flexural stiffness. The solution of this equation is:

$$\bar{w}^p_n = \frac{P}{4bD}\frac{1}{\alpha_n^2+\beta_n^2}\left(\frac{1}{\alpha_n}e^{-\alpha_n\psi}\cos\beta_n\psi + \frac{1}{\beta_n}e^{-\alpha_n\psi}\sin\beta_n\psi\right) \tag{13}$$

where:

$$\alpha_n = \sqrt{\frac{1}{2}\left[\sqrt{1+\frac{n^4\pi^4}{b^4}}+\frac{n^2\pi^2}{b^2}\right]} \tag{14}$$

and

$$\beta_n = \sqrt{\frac{1}{2}\left[\sqrt{1+\frac{n^4\pi^4}{b^4}}-\frac{n^2\pi^2}{b^2}\right]} \tag{15}$$

Substituting Equation 13 in Equation 11 and performing the summation prescribed by Equation 9 would yield the deflection at any point (x,ψ) due to a single line of interior concentrated loads. The corresponding deflection due to both lines of loads may be obtained by superposition. It is more convenient, however, to write the combined solution in terms of coordinates (x,y) instead of (x,ψ), with y being measured from the line of symmetry between the two lines of loads (corresponding to the free edge in the original system). Thus, Equation 11 becomes:

$$w^p_n(x,y) = \left[\bar{w}^p_n(|y-\bar{D}|) + \bar{w}^p_n(|y+\bar{D}|)\right]\cos\frac{n\pi x}{b} \tag{16}$$

Using Equation 13, Equation 16 may be rewritten as:

$$w^p_n(x,y) = \frac{P}{4bD}\frac{1}{\alpha_n^2+\beta_n^2}\left(\frac{1}{\alpha_n}\left[e^{-\alpha_n|y+\bar{D}|}\cos\beta_n|y+\bar{D}| + e^{-\alpha_n|y-\bar{D}|}\cos\beta_n|y-\bar{D}|\right] + \frac{1}{\beta_n}\left[e^{-\alpha_n|y+\bar{D}|}\sin\beta_n|y+\bar{D}| + e^{-\alpha_n|y-\bar{D}|}\sin\beta_n|y-\bar{D}|\right]\right)\cos\frac{n\pi x}{b} \tag{17}$$

This equation may be used to obtain the primary solution sought by performing the following summation according to Equation 9:

$$w^p(x,y) = \frac{w^p_0(x,y)}{2} + \sum_{n=1}^{\infty} w^p_n(x,y) \tag{18}$$

Note that due to symmetry, the vertical shearing forces along y=0 (free edge in original system) are identically equal to zero.

Compensative Solution

The compensative solution is introduced for the purpose of eliminating the bending moments generated along y=0 by the primary solution, without introducing any shear forces along that line. It may be determined by considering an infinite plate subjected to angular deformations $\theta(x)$ along y=0. The unknown function $\theta(x)$ may be assumed to be of the following form:

$$\theta(x) = \frac{\theta_0}{2} + \sum_{n=1}^{\infty}\theta_n\cos\frac{n\pi x}{b} \tag{19}$$

in which θ_n are coefficients to be determined. Consequently, each elemental term of the compensative solution may also be assumed to have the form:

$$w^\theta_n(x,y) = \bar{w}^\theta_n(y)\cos\frac{n\pi x}{b} \tag{20}$$

with y measured from the line of symmetry between the two line loads (i.e., the original plate edge). Proceeding as described earlier for the primary solution, the following expression for the unknown deflection function $\bar{w}^\theta_n(y)$ is obtained:

$$\bar{w}^\theta_n(y) = A_n e^{-\alpha_n y}\cos\beta_n y + B_n e^{-\alpha_n y}\sin\beta_n y \tag{21}$$

with α_n and β_n as defined by Equations 14 and 15, above. Constants A_n and B_n can be determined from the boundary conditions at y=0, i.e.,

$$\frac{dw^\theta(x,y)}{dy} = \frac{\theta(x)}{2} \tag{22}$$

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2\partial y}\right] = 0 \tag{23}$$

These lead to:

$$A_n = -\frac{\theta_n}{2(\alpha_n+\kappa_n\beta_n)} \tag{24}$$

and

$$B_n = \frac{\kappa_n\theta_n}{2(\alpha_n+\kappa_n\beta_n)} \tag{25}$$

where:

$$\kappa_n = \frac{\alpha_n(\alpha_n^2-\beta_n^2)(1-2\mu)+\beta_n}{-\beta_n(\alpha_n^2-\beta_n^2)(1-2\mu)+\alpha_n} \tag{26}$$

Having determined constants A_n and B_n, Equation 21 may be substituted in Equation 20 resulting in the following expression for the deflection due to each elemental term of the compensative angular load:

$$w_n^{\theta}(x,y) = \frac{\theta_n}{2(\alpha_n+\kappa_n\beta_n)}[-\cos\beta_n y+\kappa_n\sin\beta_n y]e^{-\alpha_n y}\cos\frac{n\pi x}{b} \tag{27}$$

The bending moments due to each elemental term are obtained by differentiation of Equation 27:

$$M_{yn}^{\theta} = -D\left(\frac{\partial^2 w_n^{\theta}}{\partial y^2}+\mu\frac{\partial^2 w_n^{\theta}}{\partial x^2}\right)$$

$$= -D\frac{\theta_n}{2(\alpha_n+\kappa_n\beta_n)}\left\{-\left[(1-\mu)\frac{n^2\pi^2}{b^2}+\kappa_n\right]\cos\beta_n y\right. \tag{28}$$

$$\left.+\left[(1-\mu)\kappa_n\frac{n^2\pi^2}{b^2}-1\right]\sin\beta_n y\right\}e^{-\alpha_n y}\cos\frac{n\pi x}{b}$$

and

$$M_{xn}^{\theta} = -D\left(\frac{\partial^2 w_n^{\theta}}{\partial x^2}+\mu\frac{\partial^2 w_n^{\theta}}{\partial y^2}\right)$$

$$= -D\frac{\theta_n}{2(\alpha_n+\kappa_n\beta_n)}\left\{\left[(1-\mu)\frac{n^2\pi^2}{b^2}-\kappa_n\mu\right]\cos\beta_n y\right. \tag{29}$$

$$\left.-\left[(1-\mu)\kappa_n\frac{n^2\pi^2}{b^2}+\mu\right]\sin\beta_n y\right\}e^{-\alpha_n y}\cos\frac{n\pi x}{b}$$

To determine coefficients, θ_n, the following equation needs to solved:

$$M_{yn}^{p}\big|_{x=0} = -D\left(\frac{\partial^2 w_n^{p}}{\partial y^2}+\mu\frac{\partial^2 w_n^{p}}{\partial x^2}\right)\Big|_{x=0} = M_{yn}^{\theta}\big|_{x=0} \tag{30}$$

This leads to:

$$\theta_n = \frac{P}{bD}\cdot\frac{\alpha_n+\kappa_n\beta_n}{(1-\mu)\frac{n^2\pi^2}{b^2}+\kappa_n}\left[-\frac{1}{\alpha_n}\left(1+\mu\frac{n^2\pi^2}{b^2}\cdot\frac{1}{\alpha_n^2+\beta_n^2}\right)\cos\beta_n a\right. \tag{31}$$

$$\left.+\frac{1}{\beta_n}\left(1-\mu\frac{n^2\pi^2}{b^2}\cdot\frac{1}{a_n^2+\beta_n^2}\right)\sin\beta_n a\right]e^{-\alpha_n a}$$

The compensative solution sought is obtained by substituting Equation 31 in Equation 27 and performing the following summation in accordance to Equation 19:

$$w^{\theta}(x,y) = \frac{w_0^{\theta}(x,y)}{2}+\sum_{n=1}^{\infty}w_n^{\theta}(x,y) \tag{32}$$

Complete Solution

The complete solution is obtained by adding the primary and compensative solutions defined by Equations 18 and 32, respectively:

$$w_c(x,y) = w^{p}(x,y)+w^{\theta}(x,y) \tag{33}$$

Bending moment and shearing force resultants may be obtained from Equation 33 by differentiation. The corresponding solution for a loaded area of finite size may be obtained from Equation 33 by integration. For example, considering a rectangular loaded area extending from χ_1 to χ_2 in the x-direction, and from $\bar{D}_1$ to $\bar{D}_2$ in the y-direction, the deflection at any point w(x,y) may be written as:

$$w(x,y) = \frac{1}{A}\int_{\chi_1}^{\chi_2}\int_{\bar{D}_1}^{\bar{D}_2} w_c(|x-\chi|,y,\bar{D})\,d\chi\,d\bar{D} \tag{34}$$

where A is the area of the load:

$$A = |\chi_1-\chi_2|\cdot|\bar{D}_1-\bar{D}_2| \tag{35}$$

The main advantage of Korenev's solution is that unlike the one presented by Pickett, et al. *(44)*, it does not contain improper integrals and may, therefore, be incorporated easily into a personal computer spreadsheet, or a mathematical software package. As an illustration, Korenev's MCL was applied in this study to the load transfer problem arising when two adjacent semi-infinite plates resting on a DL foundation are connected by dowel bars, assumed to act as pure-shear load transfer devices. When an edge load is applied to one of the plates, the transferred load by each dowel, P_i, is proportional to the difference of the deflections (w_i-w_i'), arising at the same dowel location on the loaded and unloaded sides of the joint, respectively, i.e.,

$$P_i = B\left[w_i-w_i'\right] \tag{36}$$

Parameter B is an effective stiffness of the joint to be defined later. The unknown dowel shear forces, P_i, can be determined by solving a system of linear equations, each pertaining to one of the n dowels, e.g.,:

$$\left[2\sum_{j=1}^{n}\gamma_{ij}+\frac{1}{B}\delta_{ij}\right]P_j = \Delta_i^{f} \tag{37}$$

In Equation 37, γ_{ij} is the free-edge deflection at location i due to a unit concentrated load at location j, determined using Korenev's MCL; δ_{ij} is the Kronecker delta; and Δ_i^{f} is the free-edge

deflection at location i due to the actual applied edge load, also determined using the MCL. Comparison of this solution with finite element results presented by Ioannides and Korovesis *(45)* shows excellent agreement when B is set equal to the composite (springs-in-series) shear stiffness of the supported dowel. Thus, an improved method for calculating the load transferred by the critical dowel, P_c, is suggested:

$$P_c = B\frac{(1-LTE_\delta)}{(1+LTE_\delta)}\Delta_c^f \qquad (38)$$

where Δ_c^f is the maximum edge deflection obtained using Westergaard's equation, and LTE_δ is the deflection load transfer efficiency. The method presented here has been implemented using commercial software *MATHEMATICA (46)*, and is much less tedious and time consuming than the finite element method.

Although Korenev's approach is very powerful when applied to the DL foundation, it is not as effective when other foundation models are considered. The MCL was subsequently elaborated further by Serebrjanyi *(47)*, Travush *(48; 49; 50; 51)*, and others. Serebrjanyi presents extensive tables for the solution of two semi-infinite plates resting on an ES layer of finite or infinite thickness, subjected to a single concentrated edge load. Travush extended the method to the case of a grid of rectangular plates resting on a generalized linear elastic foundation, with a variety of edge boundary conditions. Because of their broader applicability, Travush's contributions are highlighted below. Both Serebrjanyi and Travush employ Fourier integrals instead of Fourier series used by Korenev.

Travush *(48)* proposes that two plates having a common edge along the line $y=y_0$ may be treated as one continuous plate, subjected to the following generalized edge compensative load:

$$p(x) = L_1[A_1(x)\ \delta(y-y_0)] + L_2[A_2(x)\ \delta(y-y_0)] \qquad (39)$$

where L_1 and L_2 are boundary condition operators; $A_1(x)$ and $A_2(x)$ are unknown functions which satisfy the boundary conditions; and $\delta(y-t)$ is the Dirak Delta function. Considering, for example, moment and shear force boundary conditions:

$$L_1 = \frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2} \qquad (40)$$

$$L_2 = \frac{\partial^3}{\partial y^3} + (2-\mu)\frac{\partial^3}{\partial x^2 \partial y} \qquad (41)$$

Therefore, for a free edge, $A_1(x)$ and $A_2(x)$ should satisfy the following conditions:

$$\begin{aligned} L_1 w(x,y) &= 0 \qquad at\ y=y_0 \\ L_2 w(x,y) &= 0 \qquad at\ y=y_0 \end{aligned} \qquad (42)$$

Travush *(48; 50)* shows that for the case of two semi-infinite plates, unknown functions, $A_1(x)$ and $A_2(x)$, may be determined by solving a system of two linear equations. For a grid of rectangular plates, solution of a system of Fredholm integral equations is required.

As an illustration, consider two adjacent semi-infinite plates resting on an ES foundation, subjected to a single load, P. For this case, $A_2(x) = 0$, and the deflection distribution of the loaded plate can be presented in the form *(47)*:

$$w(x,y) = w^p(x,y) + \frac{P\ell^2}{4D\pi^2}\int_{+\infty}^{-\infty} \int_{+\infty}^{-\infty} \frac{A_1(\eta)\,(\xi^2+\mu\eta^2)\exp(-i\eta x - i\xi y)}{\sqrt{\eta^2+\xi^2}\left[(\eta^2+\xi^2)^{\frac{3}{2}}+1\right]} d\eta\, d\xi \qquad (43)$$

where $w^p(x,y)$ is the deflection corresponding to the primary (interior loading) solution, and:

$$A_1(\eta) = \frac{\int_{-\infty}^{+\infty} \frac{(\xi^2+\mu\eta^2)\,Q(\eta,\xi)}{\sqrt{\eta^2+\xi^2}\left[(\eta^2+\xi^2)^{\frac{3}{2}}+1\right]} d\xi}{\frac{\pi}{2}(3+\mu)(1-\mu)|\eta| + \int_{-\infty}^{+\infty} \frac{(\xi^2+\mu\eta^2)^2}{(\eta^2+\xi^2)^2\left[(\eta^2+\xi^2)^{\frac{3}{2}}+1\right]} d\xi} \qquad (44)$$

in which $Q(\eta,\xi)$ is the Fourier transform of the applied load P:

$$Q(\eta,\xi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x,y)\ \exp(i\eta x + i\xi y)\ d\eta\, d\xi \qquad (45)$$

and p(x,y) is the applied pressure. Stroganova *(23)* applied Travush's method to the problem of four quarter-infinite plate resting on a ZSS and on a Repnikov foundation. A method similar to Travush's has been proposed by Tseitlin *(52; 53)*.

SINGULAR STRENGTH CRITERIA

Experimental results often seem to suggest that a "pavement fails at a critical condition rather than long-term fatigue" *(54)*. This possibility becomes even more important when one considers the overloading of the nation's highways, as well as the variation in structural capacity experienced by the pavement system during the annual cycle. Thus, a "limit state" may be expected to occur when heavily overloaded trucks are applied during periods of considerable loss in pavement strength, such as during spring thaw or partial contact conditions due to a temperature differential through the thickness of the slab. Under such conditions, concrete pavement failure would be related to rapid propagation of cracks rather than to their first appearance.

During the last two decades, significant achievements have been accomplished in the application of fracture mechanics to the problem of crack propagation in concrete structures. Noteworthy among the numerous contributions are those by Bažant and his colleagues *(55)*, particularly their development of nonlinear fracture mechanics concepts. Application of such methods to pavement analysis should be very beneficial, and it may be expected that this will be a major direction to be pursued in the not too distant future. At the present time, however, the pursuit of a simpler yet reliable approach appears more realistic. The Stress Gradient Criteria of

Strength (SGCS) for quasi-brittle materials, first proposed by Kharlab and Minin *(56)* and later elaborated by Kharlab *(57; 58)*, offer one such option. Rather than focusing on the micro-phenomena occurring within the concrete matrix, as done, for example, using linear and nonlinear fracture mechanics and plasticity theory, Kharlab's approach is to recognize the complex nature of this locally inhomogeneous material, and seek simpler yet adequately precise solutions by considering its macro-behavior. This is particularly suited for pavement analysis, since the conventional use of linear elasticity for primary response calculation does not justify the use of elaborate failure criteria. Solutions obtained by Kharlab to date have been shown to be good approximations of those from more complex formulae derived using fracture mechanics.

The formulation of the SGCS is based on the experimental observation that local strength of the material is higher where elastic stress distribution is more non-uniform. The observation that infinite linear elastic stresses, such as those predicted at the tip of a crack, or under the edge of a plate on an ES foundation, are necessarily fictitious, prompted Kharlab to seek an alternative set of strength criteria. His SGCS are based on the hypothesis that a material may not fail under theoretically predicted high (or even infinite) stresses, if these are sufficiently localized in nature. Stated in another way, this hypothesis recognizes the significance of the stress gradient, a factor recognized in the West, as well *(59)*. Thus, for those cases where linear elasticity predicts infinite stresses, singular criteria of strength must be developed for the assessment of structural adequacy on the basis of the stress gradient.

In fracture mechanics, strength assessment usually involves an application of energy principles and results in prediction of crack propagation. The mathematics of such a formulation tends to be rather cumbersome. Kharlab's proposal is to consider not only the maximum stress value, but also the stress gradient, since it is found that when the gradient is higher so is the strength. This can perhaps explain why center-point loading of concrete beams leads to higher modulus of rupture values than third-point loading. The major focus of Kharlab's considerations is also the propagation of cracking, rather than merely the onset of cracking. The problem of crack propagation is of paramount importance in pavement analysis and design, since even newly placed PCC slabs invariably include a number of cracks developing during construction. Predictions from this approach are similar to those from fracture mechanics, even though they much easier and convenient to obtain.

Using information provided by the classical theory of elasticity, SGCS allow the estimation of strength mobilized at a regular point, i.e., at a point where the stress distribution is smooth, as well as at a singularity one (singularity kind: r^{-n}), i.e., a point where the predicted elastic stress tends to infinity. Consider, for example, a beam subjected to pure bending. The stress resultant of interest, S(x,y,z), is in this case the bending stress, $\sigma_b(z)$, predicted by classical beam theory to be:

$$S(x,y,z) = \sigma_b(z) = \frac{M z}{I} \tag{46}$$

where M is the applied bending moment, I is the cross-sectional area moment of inertia, and z is the distance from the neutral axis of the point under consideration. A common failure criterion is to assume that failure is initiated at a point where the value of the stress resultant considered, S(x,y,z), reaches the local strength of the material, R(x,y,z), i.e.,:

$$S(x,y,z) \le R(x,y,z) \tag{47}$$

According to SGCS, the local strength, R(x,y,z), is not merely a material characteristic but also depends on the distribution of the stress resultant used in the formulation of the failure criterion, and more specifically on the stress gradient. The mathematical expression of this assertion is:

$$R(x,y,z) = R_0 \left(1 + \delta \frac{|\nabla S(x,y,z)|}{S(x,y,z)}\right)^m \tag{48}$$

where R_0 is the local strength mobilized under a uniform stress state (a constant), δ is a material parameter with dimension of length, and

$$|\nabla S| = \sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2} \tag{49}$$

Exponent m is set to 1 for a regular point, and to the order, n, of the singularity kind for a singularity point. For a regular point in the beam bending example, m=1 and:

$$|\nabla S| = \frac{\partial S}{\partial z} = \frac{M}{I} \tag{50}$$

and

$$\frac{|\nabla S|}{S} = \frac{1}{z} \tag{51}$$

Therefore,

$$R = R_0 \left[1 + \frac{\delta}{z}\right] \tag{52}$$

in which R_0 denotes the strength of the material determined from uniaxial tensile tests. As M increases to failure, Equations 46 and 52 will first become equal at z=h/2, where h is the beam thickness. Thus, the SGCS for this case is:

$$\sigma_b\left(\frac{h}{2}\right) \le R_0 \left(1 + \frac{2\delta}{h}\right) \tag{53}$$

Equation 53 explains the dependence on specimen size of the concrete modulus of rupture determined from beam tests, a fact known from experimental works in both Russia and in the West *(60)*. Kharlab compared his theoretical result for this case to experimental data obtained by Karavaev *(61)* and found excellent agreement. Thus, he was able to determine parameter δ from tests conducted on beams of various cross-sections subjected to pure bending.

Note that in general S(x,y,z) may be a function of the principal stresses, σ_1, σ_2, σ_3, determined using the theory of elasticity. Commonly used forms of S(x,y,z) include the first stress invariant, θ; the deviator stress, σ_D; stress coordinates p and q *(62)*; etc.

For a singularity point, Kharlab *(58)* showed that Equation 47 is equivalent to the following condition:

$$S|_{r=n\delta} = R_0 \tag{54}$$

in which r is the distance of the point under consideration from the singularity point. As an illustration of such an application of SGCS, consider an elastic body containing a crack of size 2a, under a uniform tensile stress p. The normal stress near the crack tip may be expressed in polar coordinates as follows *(63)*:

$$S(r,\theta) = K_I \frac{1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{55}$$

in which K_I is the stress intensity factor for mode I (opening crack):

$$K_I = p\sqrt{\pi a} \tag{56}$$

For this problem in linear fracture mechanics θ may be set to zero, and n is equal to 1/2, as indicated by Equation 55. Substituting Equations 55 and 56 into Equation 54 yields the following condition for no crack propagation:

$$p \le \sqrt{\frac{\delta}{a}}\, R_0 \tag{57}$$

where R_0 is the material strength under uniform tension. The corresponding condition in linear fracture mechanics would be *(55)*:

$$K_I \le K_{IC} \tag{58}$$

where K_{IC} is the fracture toughness, i.e., the critical value of the stress intensity factor, K_I. Comparing Equations 56 and 57, the following expression for the relationship between K_{IC} and δ is obtained:

$$K_{IC} = R_0 \sqrt{\pi\delta} \tag{59}$$

Two additional results presented by Kharlab *(58)* will be reproduced here without any comments. These pertain to loading of an elastic half-space. The ultimate point load, P_{ult}, on such a medium is:

$$P_{ult} = \frac{16\pi\delta^2}{(1-\mu)(1-2\mu)+6\mu}\, R_0^{(+)} \tag{60}$$

For the case of a rigid punch, the ultimate load is given be:

$$P_{ult} = 2\sqrt{\frac{\delta}{a}}\, R_0^{(-)}\, \pi a^2 \tag{61}$$

In these expressions, a is the radius of the punch, μ is the Poisson's ratio of the half-space, and $R_0^{(+)}$ and $R_0^{(-)}$ are the half-space material strengths for uniform tension and compression, respectively.

Kharlab's proposals may be incorporated in pavement analysis and design, where the assumption that pavements fail in fatigue is often used, despite the fact that this failure mechanism is not well understood. In one sense, fatigue implies that cracks propagate very rapidly, and Kharlab's simple SGCS may help determine if this will happen. Therefore, it may be suggested that δ may depend on the number of load repetitions, N, since δ is exclusively a material and not a structural system property, is independent of specimen size and may reflect accurately the fatigue properties of concrete. If this hypothesis were correct, parameter δ would be estimated from laboratory tests performed on one particular structural system (e.g., on a simply supported beam under two point loads), but subsequently be applied to another system in the field (e.g., continuously supported slabs on a layered foundation loaded by arbitrary multi-wheel loads). Whereas the number of repetitions in the field (slab) will generally not be equal to the corresponding number in the laboratory test (beam), the variation of δ with N would remain the same. This would result in a more mechanistic means of accounting for specimen size effects. The validity of this hypothesis is currently being experimentally tested at the University of Illinois.

CONCLUSIONS

This Paper presents a review of several advances in the state-of-knowledge in the former Soviet Union over the last thirty years, which may be applicable to pavement analysis and design in the U.S. today. About forty articles and books in Russian have been identified and studied, and a few have even been translated into English. Although the former Soviet Union would hardly be recognized as a leader in pavement technology, examination of a number of theoretical works that have recently become accessible to the West confirms that significant breakthroughs can be achieved if these were developed to the point where they became adequately practical. Suggestions from the East may be instrumental in developing improved analytical and numerical solutions suitable for inclusion in an efficient design algorithm. In particular, four issues directly related to problems encountered in concrete pavement analysis and design are examined in this Paper: the formulation of an improved subgrade model which would address the limitations of the conventional dense liquid and elastic solid idealizations; the development of a forward and back-calculation procedure capable of accommodating a multi-layered concrete pavement system resting on a DL or ZSS foundation; the reconsideration of some pervasive edge loading problems; and the application of a simplified approach to pavement failure, capable of reproducing more complex fracture mechanics solutions.

The ZSS foundation model is an effective yet practical means of combining the advantages of both the DL and ES subgrades in a manner that reproduces real soil behavior more faithfully than either of the two extreme conventional idealizations. In this study, a solution in the form of an infinite integral (Equation 2) has been obtained for a circular interior load on a plate resting on a ZSS

foundation. Although interior loading is often not critical, such research holds significant potential in the pursuit of a "unified" pavement analysis and design approach, such as pursued by the FAA. Furthermore, solutions for interior loading are elementary and may eventually be extended to the more critical edge and corner loading conditions.

The formulations presented by Korenev *(24; 7; 43)*, as well as those by Van Cauwelaert *(37)* in the West, may be incorporated into a multi-layered analysis procedure for PCC pavements similar to computer programs such as *BISAR (64)* and *CHEVRON (65)* used for the analysis of AC pavements. Such a procedure will retain the DL (or the ZSS) foundation where necessary and will eliminate the need to model relatively soft constructed layers (base, subbase, or AC overlay) using plate theory. It may be expected that the development of this new computer program will contribute significantly toward the development of the "unified" approach.

Edge loading problems which often control the design of PCC pavements may be investigated very efficiently using the Method of Compensative Loads, which treats two semi-infinite plates as a continuous (infinite) plate, loaded by the actual load, as well as by an additional fictitious line load introduced to simulate the presence of the discontinuity. The main attractive attribute of this approach is that can lead to a closed-form solution for the edge loading problem, using any chosen subgrade idealization. Such a solution would be much easier to implement in a design algorithm. Solutions presented in Soviet literature have been extended in this investigation to include load transfer effects, thus verifying conclusions of earlier finite element studies involving the DL subgrade. Incorporation of edge loading capabilities in the new procedure envisaged for multi-layered PCC pavements would constitute a major improvement of current analytical capabilities.

In parallel to such pursuits, the development of improved design algorithms for PCC pavements necessitates the definition of "limit states" responsible for rapid structural deterioration of many pavement systems, and the formulation of defenses against these, in addition to fatigue. Research conducted by Kharlab at the Institute of Civil Engineering in former Leningrad (LISI) can be very useful in this direction. His "Singular Criteria of Strength" recognize the significance of the stress gradient, can account for specimen size effects, and offer a simple yet powerful method of predicting crack propagation, without resorting to the more cumbersome formulations of linear and nonlinear fracture mechanics.

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TRANSLITERATION TABLE

А	а	a	Р	р	r
Б	б	b	С	с	s
В	в	v	Т	т	t
Г	г	g	У	у	u
Д	д	d	Ф	ф	f
Е	е	e (ye)	Х	х	kh
Ж	ж	zh	Ц	ц	ts
З	з	z	Ч	ч	ch
И	и	i	Ш	ш	sh
Й	й	i	Щ	щ	shch
К	к	k	Ъ	ъ	″
Л	л	l	Ы	ы	y
М	м	m	Ь	ь	′
Н	н	n	Э	э	e
О	о	o	Ю	ю	iu (yu)
П	п	p	Я	я	ia (ya)

METRIC CONVERSION FACTORS

1 in. = 25.4 mm
1 lb = 4.44822 N
1 psi = 6.89476 kPa
1 psi/in. = 0.27145 MN/m^3

TABLE 1 SUBGRADE MODELS CONSIDERED BY CHERKASOV *(12)*

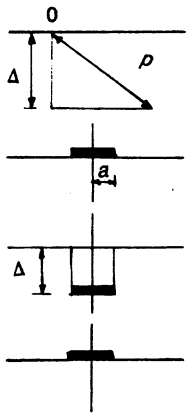
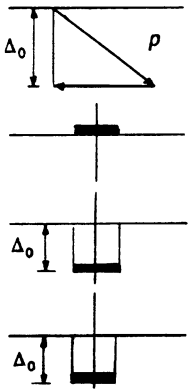
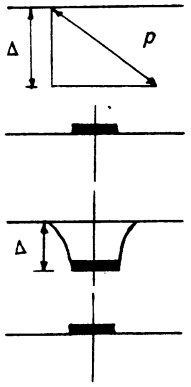
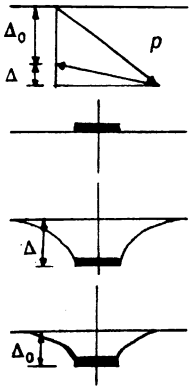
MODEL NAME	1. WINKLER'S MODEL	2. FUSS' MODEL	3. ELASTIC HALF-SPACE	4. LINEAR HALF-SPACE
Diagram of Loading and Unloading by Rigid Plate Surface Deformations: Initial State Plate Load Applied Plate Load Removed				
Deformation Character	Local, Elastic	Local, non-recoverable	Global, Elastic	Global, Elastic Global, Non-recoverable
Mechanical Parameters	Subgrade Coefficient, k	Subgrade Coefficient, k	Elastic Modulus, E_s Poisson Ratio, μ_s	Elastic Modulus, E_s Deformation Modulus, E_0 Poisson Ratios, μ_s, μ_0
Deflection Under Plate:				
Elastic	$\Delta = \frac{p}{k}$	0	$\Delta = \frac{\pi}{2} a \frac{1-\mu_s^2}{E_s} p$	$\Delta = \frac{\pi}{2} a \frac{1-\mu_s^2}{E_s} p$
Non-recoverable	0	$\Delta_0 = \frac{p}{k}$	0	$\Delta_0 = \frac{\pi}{2} a \left(\frac{1-\mu_0^2}{E_0} - \frac{1-\mu_s^2}{E_s} \right) p$
Deflection Beyond Plate:				
Elastic	0	0	$\Delta' = p\, a \frac{1-\mu_s^2}{E_s} \sin^{-1}\left(\frac{a}{r}\right)$	$\Delta' = p\, a \frac{1-\mu_s^2}{E_s} \sin^{-1}\left(\frac{a}{r}\right)$
Non-recoverable	0	0	0	$\Delta_0' = a \left(\frac{1-\mu_0^2}{E_0} - \frac{1-\mu_s^2}{E_s} \right) \sin^{-1}\left(\frac{a}{r}\right) p$

TABLE 1 *(Contd.)* SUBGRADE MODELS CONSIDERED BY CHERKASOV *(12)*

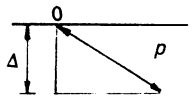
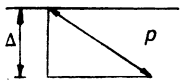
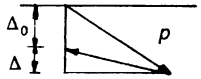
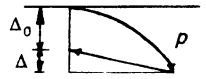
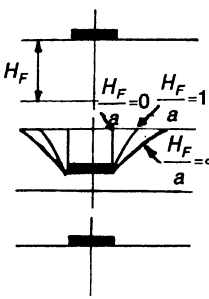
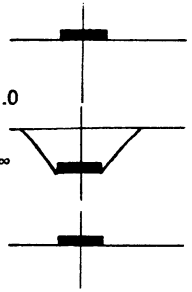
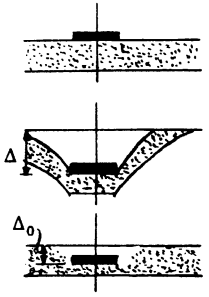
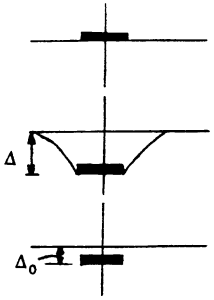
MODEL NAME	5. COMPRESSIBLE LAYER OF FINITE DEPTH	6. PASTERNAK'S MODEL	7. SINITSYN'S MODEL	8. CHERKASOV AND KLEIN MODEL
Diagram of Loading and Unloading by Rigid Plate				
Surface Deformations: Initial State; Plate Load Applied; Plate Load Removed				
Deformation Character	Global, Elastic	Global, Elastic	Local, Non-recoverable Global, Elastic	Local, Non-recoverable Global, Elastic
Mechanical Parameters	Layer thickness, H_F Elastic Modulus, E_F Poisson Ratio, μ_F	Compression Coefficient, k_1 Shear Coefficient, G	Subgrade Coefficient, k Elastic Modulus, E_S Poisson Ratio, μ_S	Hardness Number, A; Hardening Exponent, n; Elastic Modulus, E_S; Poisson Ratio, μ_S
Deflection Under Plate:				
Elastic	Egorov Formula or Shekhter Table	$\Delta = \dfrac{a\,p}{a\,k_1 + 2\sqrt{k_1 G} + 2\dfrac{G}{a}}$	$\Delta = \dfrac{\pi}{2}\,a\,\dfrac{1-\mu_s^2}{E_s}\,p$	$\Delta = \dfrac{\pi}{2}\,a\,\dfrac{1-\mu_s^2}{E_s}\,p$
Non-recoverable	0	0	$\Delta_0 = \dfrac{p}{k}$	$\Delta_0 = 2a\sqrt[n]{\dfrac{p}{A}}$
Deflection Beyond Plate:				
Elastic	Egorov Formula or Shekhter Table	$\Delta = \dfrac{ap}{ak_1 + 2\sqrt{k_1 G} + 2\dfrac{G}{a}} \exp\left[-\sqrt{\dfrac{k_1}{G}}\,r\right]$	$\Delta' = a\,\dfrac{1-\mu_s^2}{E_s}\,\sin^{-1}\left(\dfrac{a}{r}\right)p$	$\Delta' = a\,\dfrac{1-\mu_s^2}{E_s}\,\sin^{-1}\left(\dfrac{a}{r}\right)p$
Non-recoverable	0	0	0	0

TABLE 2 INFLUENCE FUNCTIONS AND THEIR FOURIER TRANSFORMS FOR VARIOUS SOIL MODELS

MODEL	INFLUENCE FUNCTION (KERNEL)	FOURIER TRANSFORM	EXPLANATIONS
Dense Liquid *(25)*	$\frac{1}{k}\,\delta^*(r)$	$\frac{1}{k}$	$\delta^*(r) = \begin{cases} 0 & \text{for } r\neq 0 \\ \lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2 k} & \text{for } r=0 \end{cases}$
Elastic Solid *(25)*	$\frac{1}{\pi C}\,\frac{1}{r}$	$\frac{2}{C}\,\frac{1}{\alpha}$	
Zhemochkin-Sinitsyn-Shtaerman *(16; 17)*	$\frac{1}{k}\,\delta^*(r) + \frac{1}{\pi C}\,\frac{1}{r}$	$\frac{2}{C}\,\frac{1}{\alpha} + \frac{1}{k}$	
Korenev *(24)*	$\frac{B}{\sqrt{r^2+\epsilon^2}}$	$\frac{2\pi B}{\alpha}\exp(-\alpha\epsilon)$	
Petrosyan *(26)*	$\frac{B}{\left(2\pi\sqrt{r^2+\epsilon^2}\right)^{1+\nu}}$	$\frac{\alpha^{\frac{\nu+1}{2}}\,\epsilon^{\frac{1-\nu}{2}}\,K_{\frac{1-\nu}{2}}\,\epsilon\alpha}{2^{\frac{3}{2}(\nu-1)}\,\pi^{\nu}\,\Gamma\left(\frac{1+\nu}{2}\right)}$	$K_{\frac{1-\nu}{2}}(\epsilon\alpha)$: Modified Bessel function of the second kind $\Gamma\left(\frac{1+\nu}{2}\right)$: Gamma function
Klein *(27)*	$\frac{B}{\pi\, r^{n+1}}$	$2^{1-n}\,B\,(\alpha)^{n-1}\frac{\Gamma\left(\frac{1-n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$ $\left(\frac{1}{2}<n<1\right)$	α, ν, ϵ, n : dummy variables r : radial distance

TABLE 3 MAX. STRESS (psi) AT BOTTOM OF LAYER 1 IN THREE-LAYER SYSTEM

h_2 (in.)	LAYERS-ON-DL *(This study)*	"CLOSED-FORM" *(41)*	MODIFIED TOTSKY *(This study)*	ORIGINAL TOTSKY *(40)*
0	267.7	261.0*	261.2	261.2
4	262.1	259.9	257.0	256.9
5	257.9	258.9	253.5	253.2
6	252.5	257.5	248.8	248.2
7	246.0	255.5	243.1	241.9
8	238.7	252.9	236.5	234.7
9	230.9	249.9	229.2	226.6
10	222.7	246.3	221.5	218.0

Notes: E_1=4 Mpsi; E_2=400 ksi; h_1= 7 in.; μ_1=0.15; μ_2=0.2; k=100 psi/in.
P=10 kips; p= 90 psi

*For this case, "CLOSED-FORM" gives the same result as Westergaard's "Ordinary Theory"; the corresponding "Special Theory" prediction is 268.7 psi.