THE PASTERNAK FOUNDATION

An Attractive Alternative for the Winkler Foundation

A.C. Pronk
Road and Hydraulic Engineering Division
Directorate-General of Public Works
and Water Management; The Netherlands

SUMMARY

In this paper the Pasternak Foundation model is described. In this two parameter model the reaction of the foundation is determined by a vertical spring constant k (the modulus of subgrade reaction in the Winkler Foundation) in combination with a parameter k^* which can be seen as a horizontal linkage of the vertical springs (comparable to the function of the Poisson ratio in the Burmister Model). The influence of the constant k^* on the response is larger than the influence of the Poisson ratio in the Burmister Model on the response. This increases the flexibility of this model in matching the observed and calculated responses of a construction.

For the calculations of the deflections and stresses in the Pasternak and Westergaard Model we used the Hankel-Bessel transformation. Only the case of interior loading has been considered. Solutions are given for the deflection bowls and the maximum bending stress for a (thin) plate, resting without friction on the foundation. An overview of the Hankel-Bessel transforms is given for other types of load distributions and subgrade reactions.

It is shown that with the aid of the Pasternak Foundation a large range of possible subgrade reactions can be simulated. This range includes the normal Westergaard Model ($k^* = 0$) as well as the extreme caseof the Westergaard Supplementary Theory, in which the subgrade reaction is assumed to be that of a linear elastic halfspace (Hogg's / Jeuffroy Model). Even more 'damped' reactions than the elastic halfspace can be simulated with this model. Regarding the larger flexibility of this simple two parameter model, the Pasternak Foundation forms an attractive alternative for the Winkler Foundation.

Finally, aspects of interpretation methods are discussed for the evaluation of deflection measurements on the several types of foundations.

Abbreviations

a	: Radius of load plate	[m]
bei(x), ber(x), kei(x), ker(x)	: Kelvin functions	
h	: Layer thickness of plate	[m]
k	: Modulus of subgrade reaction	[N/m³]
k^*	: Horizontal linkage parameter	[N/m]
$q_v(r)$	: reaction of subgrade (foundation)	[Pa]
r	: Horizontal distance to load center	[m]
D	: Dimensionless parameter = L_p/L	[m/m]
E	: Youngs' modulus of plate	[Pa]
E_o	: Youngs' modulus of subgrade	[Pa]
$I_o(x)$, $I_1(x)$, $K_o(x)$, $K_1(x)$	: Modified Bessel functions	
K(x)	: Hankel transform of the subgrade reaction	
L	: Characteristic length parameter	[m]
L_o	: Characteristic length parameter	[m]
L_s	: Length parameter	[m]
L_p	: Length parameter	[m]
N	: Plate stiffness = $(E\ h^3)/(12(1-\upsilon^2))$	[Nm]
P	: Total load on the plate	[N]
$P_o(x)$	: Hankel transform of a load distribution	[Pa]
R	: Radius of a finite plate	[m]
Q_m	: Deflection ratio $W(r)/W_o$; r = m × a	[m/m]
Q_n	: Ratio ratio $W_o(a_n)/W_o(a)$; a_n = n × a	[m/m]
V	: Volume of deflection bowl	[m³]
W_o	: Center deflection	[m]
W(r)	: Deflection at distance r	[m]
Z	: Parameter (Supplementary Theory)	[-]
α	: Parameter (Supplementary Theory)	[-]
υ	: Poisson ratio of plate	[m/m]
υ_o	: Poisson ratio of foundation	[m/m]
$\sigma(r)$	: Stress distribution on the plate	[Pa]
σ_o	: Constant stress on load plate = $P/(\pi\ a^2)$	[Pa]
$\sigma_r(r)$	: Radial stress (bottom of the plate)	[Pa]
$\sigma_t(r)$	: Tangential stress (bottom of the plate)	[Pa]
$\sigma_v(r)$	: Vertical stress at top of subgrade	[Pa]
Δ.	: Laplace operator	[1/m²]
$\Delta W(r)$	: Deflection correction term	[m]
$\Delta\sigma r$	: Stress correction term	[Pa]

1. Introduction

In the design of airfield pavements the mechanical model of Westergaard (1) is often used to describe the response of the construction under loading conditions.
In the Westergaard Model the reaction of the subgrade is modelled as a Winkler Foundation ('unlinked spring system' with spring constant k). The modulus of the subgrade reaction k is defined as:

$$k = \frac{\sigma_v(r)}{W(r)} = \frac{\int_0^{2\pi}\int_0^{\infty}\sigma_v(r)\, r\, dr\, d\varphi}{\int_0^{2\pi}\int_0^{\infty}W(r)\, r\, dr\, d\varphi} = \frac{P}{V} \qquad (1)$$

If the modeling of the construction is correct the value for k can be determined by plate bearing or falling weight deflections (FWD) tests independent of the radius a of the loading plate. However, in practice the k values will in many cases depend on the radius a of the plate. Westergaard himself discerned this gap between the model and reality (2). He introduced a correction term $\Delta W(r)$ for the deflection $W_r(r)$ which was determined by three parameters α, Z, and L_s (equation 2).

$$\Delta W(r) = -Z W_o \left(1 - \frac{r^2}{L_s^2}\right)^{\alpha} \quad \text{for } r < L_s \qquad (2)$$

$$\Delta W(r) = 0 \quad \text{for } r > L_s$$

This approach is known as the Supplementary Theory of Westergaard. The parameter α was arbitrarily fixed at a value of 10. The corresponding correction term for the maximal bending stress in the interior case is given for that case by:

$$\Delta\sigma_r(0) = -\frac{120\, L^4\, k\, (1+\nu)}{L_s^2\, h^2}\, Z\, W_o \qquad (3)$$

Westergaard claimed that for Z = 0.3906 and L_s = 5 L this corrected model is equivalent to the Hogg/Jeuffroy Model (3) (see appendix I). That is to say the reaction of the subgrade will be the same as that of a linear elastic half space. This special case will be denoted as the extreme case of the Supplementary Theory.
The determinations of the values for Z and L_s are probably based on a comparison of the deflections due to a pointload (Herz-Hogg model) instead of a circular load (Westergaard-Jeuffroy model) (4). For large values of a/L Westergaard recommended that the following value should be taken: Z = 0.3906 × W_o(Westergaard)/ W_o(Herz).

In pavement design methods (5) nowadays the linear elastic layer theory of Burmister is often used. However the simple model of a (thin) plate resting on a foundation still has advantages, especially in the design of concrete airfield pavements. But often the model assumed for the foundation in the original Westergaard Model is not in accordance with reality. To overcome this gap one may use the correction term as suggested by Westergaard. This correction demands the determination of the parameters k, α, Z and L_s for the reaction of the foundation, which can hardly be done by deflection measurements only. Another approach is the adoption of the Pasternak foundation for the response of the foundation.

2. Pasternak Foundation

Besides the Burmister model the Pasternak Foundation is a well-known two parameter model (6).
Instead of introducing an arbitrary correction term in order to match the observed deflection bowl (as done by Westergaard) the Pasternak Model starts with a theoretical description for the response of the foundation. The reaction $q_v(r)$ of the foundation in the Pasternak Model is given by equation 4:

$$q_v(r) = k\, W(r) - k^*\, \Delta.W(r) \qquad (4)$$

This type of foundation can be considered as a linked spring system. The parameter k^* can be seen as a form of damping. The deflection ('of the spring') under the load will be less because part of the load is horizontal spread ('to the next springs'). This effect was noticed by Westergaard in practice, which induced him to introduce the correction term $\Delta W(r)$ in order to get a better match between theory and practice.

3. Integral Solutions

For an axial symmetrically loaded system the solutions for the deflections and bending stresses can easily be presented in a closed integral form using the so called Hankel-Bessel Transforms (4,7,8). An overview of the transforms for several types of foundation response and contact stress distributions is given in Appendix I. The differential equation for a (thin) plate resting on a foundation is given by equation 5:

$$N\, \Delta.(\Delta.W(r)) = \sigma(r) - q_v(r) \qquad (5)$$

For a Winkler Foundation the reaction of the subgrade $q_v(r)$ equals k.W(r) and for a Pasternak Foundation the reaction $q_v(r)$ is given by equation 4. For a circular load (σ_o) with radius a, the deflection W(r) is given by equation 6 for the Westergaard Model (4):

$$W(r) = \frac{P}{\pi a k L} \int_0^{\infty} \frac{J_1(x\frac{a}{L})\ J_o(x\frac{r}{L})}{1 + x^4}\ dx \qquad (6)$$

Equation 6 can be transformed into Kelvin functions by the splitting up of the denominator $1 + x^4$ according to equation 7.

$$\frac{1}{1+x^4} = 1 - \frac{x^2}{2(x^2-i)} - \frac{x^2}{2(x^2+i)} \qquad (7)$$

$$with\ \ i = \sqrt{(-1)}$$

The obtained integrals can be written in the form of modified Bessel functions (9) which will eventually lead to equations 8 and 9:

$$W(r) = \frac{P}{\pi a k L} \times$$

$$\left[\frac{L}{a} + \frac{d}{dy} ker(y)\ ber(\frac{r}{L}) - \frac{d}{dy} kei(y)\ bei(\frac{r}{L})\right]$$

$$for\ r < a\ ,\ \ and \qquad (8)$$

$$W(r) = \frac{P}{\pi\ a\ k\ L} \times$$

$$\left[\frac{d}{dy} ber(y)\ ker(\frac{r}{L}) - \frac{d}{dy} bei(y)\ kei(\frac{r}{L})\right]$$

$$for\ r > a\ \ and\ \ with\ y = \frac{a}{L} \qquad (9)$$

For r = 0 the first order approximation of these Kelvin functions leads to the more familiar expression:

$$W_o = W(0) = \frac{P\ L^3}{\pi\ N\ a}\left[\frac{d}{dx} ker(x)\,|_{x=\frac{a}{L}} + \frac{L}{a}\right] \qquad (10)$$

$$\approx \frac{P}{8\,k\,L^2}\left[1 + \frac{a^2}{2\pi\ L^2}\left\{\ln\left(\frac{a}{2L}\right) + \gamma - 1{,}25\right\}\right]$$

The bending stress in the Westergaard model is given by equation 11:

$$\sigma_r(r) = \frac{3\,(1+\nu)\ P\ L}{\pi\ h^2\ a} \times \int_0^{\infty} \frac{x^2\ J_1(x\frac{a}{L})\ J_o(x\frac{r}{L})}{1 + x^4}\ dx \qquad (11)$$

For r = 0 the integral can be reduced to equation 12 of which the first order approximation is more familiar:

$$\sigma_r(0) = \frac{3\,P\,L\,(1+\nu)}{\pi h^2\ a} \frac{d}{dx} kei(x)\,|_{x=\frac{a}{L}} \Rightarrow$$

$$\sigma_r(0) \approx \frac{3\,P\,(1+\nu)}{2\pi\ h^2} \times \left[\ln\left(\frac{2.L}{a}\right) - \gamma + 0{,}5 + \frac{\pi\ a^2}{32\,L^2}\right] \qquad (12)$$

In case of the Supplementary Theory (2) a correction term of $-10\pi Z \times L^2/L_s^2$ (= - 0.4908 for Z = 0.3906 and L_s = 5 L) is added between the brackets in equation 12. Therefore the bending stress in the Supplementary Theory will not be a constant percentage of the bending stress in equation 12 but will depend on the ratio a/L.
If the foundation reacts as a Pasternak Foundation the deflection W(r) due to a circular load is represented by:

$$W(r) = \frac{P}{\pi\,a\ k\ L} \times \int_0^{\infty} \left(\frac{J_1(x\frac{a}{L})\ J_o(x\frac{r}{L})}{1 + D^2\ x^2 + x^4}\right) dx \qquad (13)$$

with the substitution $D^2 = L_p^2 / L^2 = (k^*/k) \times \sqrt{(k/N)}$.

As can be seen a large similarity exists between equation 6 and equation 13 for the deflection in the Westergaard Model. In fact equation 6 can be seen as a special case ($k^* = 0$) of the more general equation 13. The integral solution for the maximal bending stress under the load is represented by equation 14:

$$\sigma_r(0) = \frac{3\,(1+\nu)\ P\ L}{\pi h^2\ a} \times \int_0^{\infty} \frac{x^2\ J_1(x\frac{a}{L})}{1 + D^2\ x^2 + x^4}\ dx \qquad (14)$$

For $k^*=0$ (Westergaard Model) equation 11 is obtained. Because of the extra 'damping' term in the denominator of equations 13 and 14 it is clear that the bending stress and deflection in the Pasternak model will be smaller than the related stress and deflection in the Westergaard Model.

4. Calculations

4.1 General The integral solutions for the deflections and the bending stresses have been calculated for both the Westergaard Model as for the Pasternak Model. The case D = 0 equals the Herz Model (if a point load is used) and the Westergaard Model (if a circular load is used).

4.2 Foundation In contrast to the Winkler Foundation the deflections outside the load plate will

not be zero in the Pasternak Model. The integral solution is given by equation 15:

$$W(r) = \frac{P}{\pi a\, k\, L_p} \times$$

$$\int_0^{\infty} \frac{J_1(x\frac{a}{L_p})\; J_o(x\frac{r}{L_p})}{1+x^2}\, dx =$$

$$= \frac{P}{\pi a^2 k}\left[1 - \frac{a}{L_p}\, K_1(\frac{a}{L_p})\; I_o(\frac{r}{L_p})\right] \quad for\ r \le a$$

$$= \frac{P}{\pi a^2 k}\left[\frac{a}{L_p}\, I_1(\frac{a}{L_p})\; K_o(\frac{r}{L_p})\right] \quad for\ r > a$$

(15)

with $L_p = \sqrt{(k^*/k)}$, and $K_{0,1}$ and $I_{0,1}$ are modified Bessel functions.

For r = 0 the function $I_o(0)$ equals 1. In figure 1 the quantity $W_o \times k/\sigma_o = F\{x\} = 1 - (a/L_p) \times K_1(a/L_p)$ is given as a function of (a/L_p).
For $L_p \to 0$ (Winkler Foundation) the functions $K_1(a/L_p) / L_p$ and $I_1(a/L_p) \times K_o(r/L_p) / L_p \to 0$ in accordance with the reaction for a Winkler Foundation. At the other hand if $L_p \to \infty$ the foundation will react as a rigid boundary with $W(r) \equiv 0$.

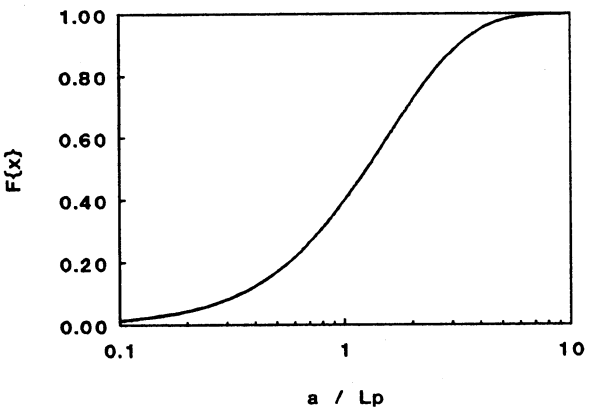


Figure 1: The function $F\{x\}$: $1 - (a/L_p) \times K_1(a/L_p)$

4.3 Thin plate resting on a foundation In case of a thin plate resting on a Pasternak Foundation it is possible to rewrite the integral solutions in the form of the real and imaginary parts of modified Bessel functions I and K. In figure 2 the relative deflection W_o with respect to the central deflection in the Westergaard model is given as a function of the parameter D for three values of the parameter a/L. It turns out that the parameter a/L has nearly no influence.

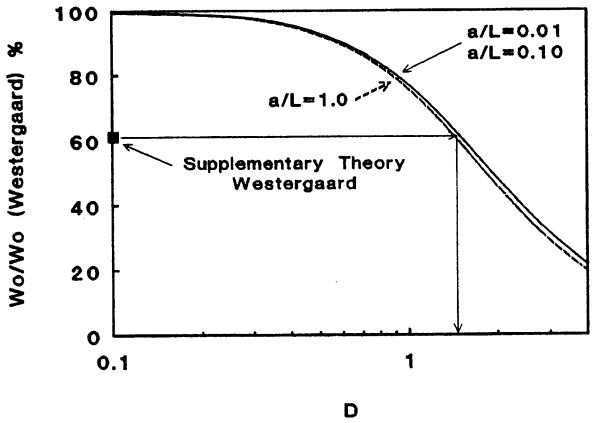


Figure 2: The ratio of the center deflection in the Pasternak Model and the Westergaard Model as a function of the parameter D for three values of a/L.

In the Supplementary Theory of Westergaard, in the extreme case in which the reaction of the foundation resembles that of linear elastic halfspace (Hogg/Jeuffroy model), the central deflection (W_o^*) equals (1-0.3906) times the central deflection (W_o) of the original Westergaard model. This percentage (61 %) is obtained in the Pasternak model at a D value of ca. 1.5.

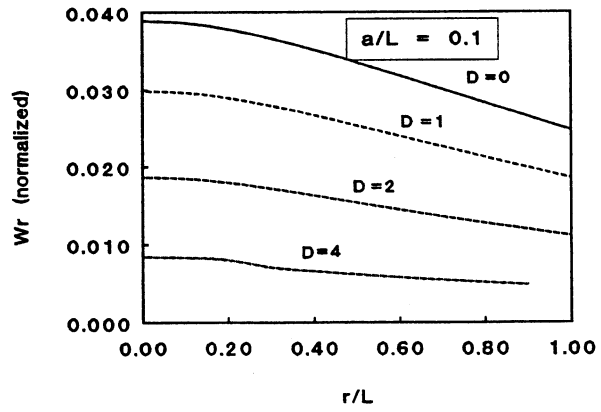


Figure 3: The normalized deflection $W(r) \times (k\,L)/(\sigma_o\, a)$ as a function of r/L for four values of D with a/L = 0.1.

In figures 3 and 4 the normalized deflections W(r) (normalized with respect to the quantity $P/(\pi\ a\ k\ L)$) are given as a function of the normalized distance r/L for four values of the parameter D and two values for the normalized radius a/L (figure 3: a/L=0.1 and figure 4: a/L=1.0).

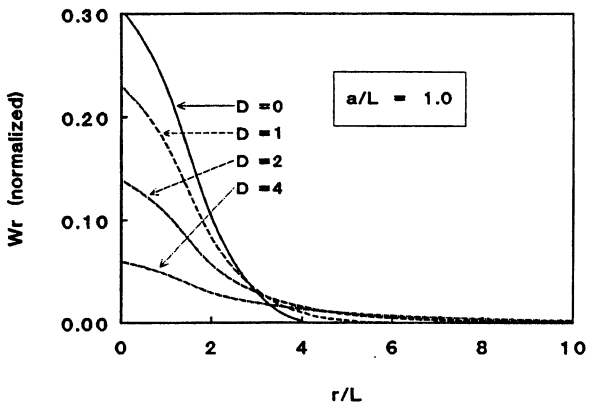


Figure 4: The normalized deflection $W(r) \times (k\ L)/(\sigma_0\ a)$ as a function of r/L for four values of D with a/L = 1.0

As shown in figure 3 the deflection W(r) will become independent of r/L for larger values of D and will tend to zero for $D \rightarrow \infty$.

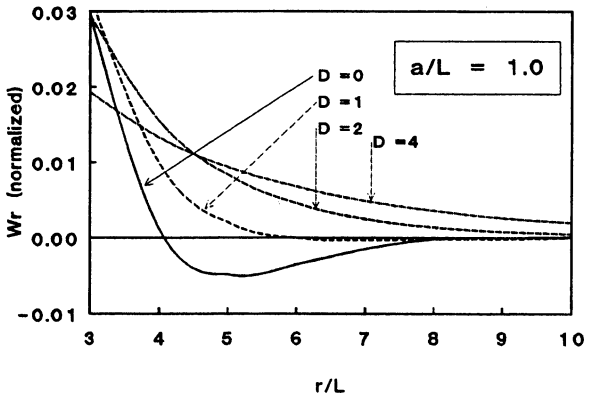


Figure 5: The normalized deflection $W(r) \times (k\ L)/(\sigma_0\ a)$ as a function of r/L for four values of D with a/L equals 1.0

In figure 5 a detail of figure 4 is given for large r/L values. It turns out that for larger r/L values the deflection in the Westergaard model (D=0) becomes negative (an upwards deflection of the plate). This may hamper the determination of the modulus of subgrade reaction, k, using the ratio of the load P and the measured deflection bowl V. If deflection measurements at several distances are used correction factors must be used as given by Ioannides (10).

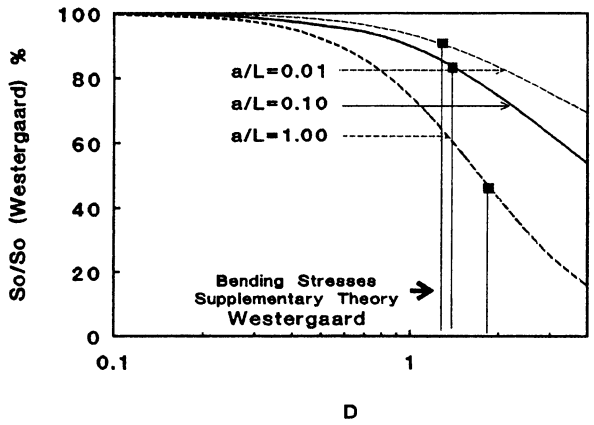


Figure 6: The ratio of the bending stress (center) in the Pasternak Model and the Westergaard Model as a function of D for three values of a/L.

In figure 6 the bending stress in the center, normalized with respect to the bending stress in the Westergaard model, is given as a function of the parameter D for three values of the parameter a/L. In contrast to the central deflection the normalized bending stress depends on the ratio a/L (comparable to the bending stress in the Supplementary Theory). Taking the value of a/L into account for the determination of the 'right' value for the constant Z the following relative percentages are found for the bending stresses in the extreme case of the Supplementary Theory (σ_0(Supplementary Theory)/ σ_0(Westergaard) $\times$ 100 %): 90.6 % for a/L = 0.01, 83.3 % for a/L = 0.10 and 46.2 % for a/L = 1.00. As can be seen in figure 6 the values for D at which the same percentages are found lie between 1.2 and 2

5. Determination of the foundation response in-situ

5.1 General Distinction should be made between measurements carried out directly on the foundation and those carried out on a (thin) plate resting on the foundation. The proposed method is based on measurements using load plates with different diameters but loaded with the same contact stress ($\sigma_0 = P/(\pi\ a^2)$). However, measurements at different stress levels are still useful in case of a stress-dependent response of the foundation.

5.2 Measurements on the foundation The ratios of the measured center deflections W_0 and the applied contact stress σ_0 are plotted on log-log scale as a function of the applied plate diameter a. The tangent of the obtained curve gives an indication of the type of foundation. In case of a Winkler foundation a horizontal

line will be obtained while in case of a Burmister foundation a straight line with a tangent of +1 will be obtained. However, in practice it is often found that the tangent of the curve decreases with increasing plate diameters. The same decrease is also observed for a Pasternak foundation.

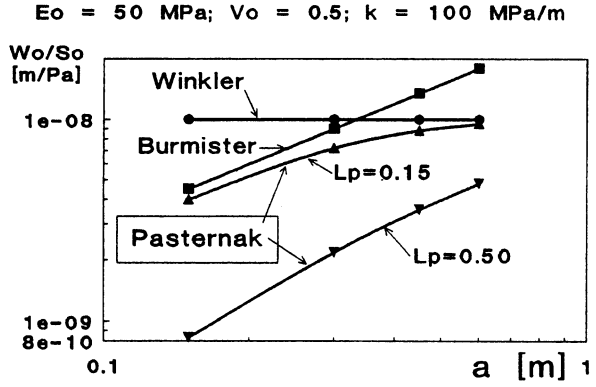


Figure 7: The normalized deflection W_o/σ_o as a function of the applied plate diameter a for three types of foundation. Measurements carried out directly on the foundation.

In figure 7 theoretical curves are given for the three types of foundation. In case of a Pasternak foundation with a L_p value of 0.15 the curve changes from a Burmister response for small values of the plate diameter a to a Winkler response for larger values of a.

5.3 Measurements on the plate - Winkler foundation: If a plate rests on the foundation, a direct determination of the modulus of subgrade reaction k will only be possible in case of a Winkler foundation. This direct determination can be performed by using the ratio of the total applied load P and the volume of the measured deflection bowl V according to equation 1. We will denominate this approach as **approach 1**. However, in practice the deflection bowl will be determined by a limited number of deflection points. As shown in figure 5 the deflection will be negative (upwards) for large values of the distance r. This might hamper a correct determination of the volume V based on a limited number of deflection points.
Besides the direct determination described above two other approaches are feasible:

approach 2: Measurements with different plate diameters a_n but with the same contact stress σ_o, and
approach 3: A curve fitting on the deflection bowl using equation 6.

These two last approaches will give values for k and L. In approach 2 it is advised to chose the plate diameters a_n according to $a_n = n \times a_1$. The ratio Q_n of the central deflection W_o for the plate diameter a_1 and the central deflection for the plate diameter a_n is given by equation 16.

$$Q_n = \frac{\int_0^\infty \frac{J_1(x\frac{a_1}{L})}{1+x^4}\,dx}{n\int_0^\infty \frac{J_1(x\,n\frac{a_1}{L})}{1+x^4}\,dx} \qquad (16)$$

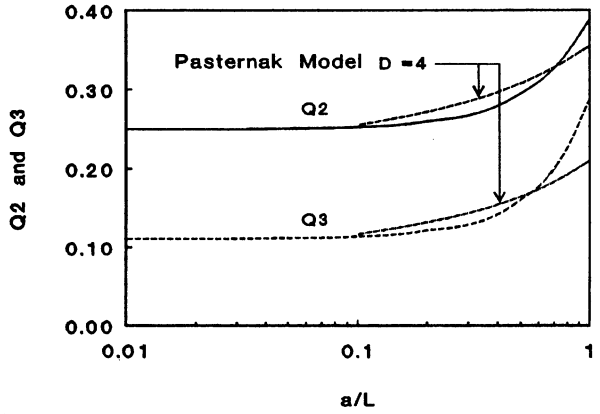


Figure 8: The ratio of the central deflections in the Westergaard Model as a function of a/L

In figure 8 the ratios Q_n for n = 2 and n = 3 are given as a function of the parameter a/L. It appears that for values of a/L smaller than 0.1 it will be not possible to determine a/L accurately. Therefore we have to use large plate diameters. The same phenomenon occurs in approach 3 if the ratio Q_m of the deflection W_o and the deflection W(r) is used ($r = m \times a_1$).
In figure 9 the ratio Q_m is given as a function of the parameter a/L for three values of m. Only for large values of m it might be possible to determine the parameter L for small a/L values.

- Burmister foundation: If a thin plate rests on a Burmister foundation (Hogg/Jeuffroy model), the subgrade modulus E_o can be estimated using deflection measurements W(r) at large distances r if a value for the Poisson ratio υ_o is adopted (equation 17:

$$W(r) \approx \frac{(1-\nu_o^2)\,P}{\pi\, r\, E_o} \qquad (17)$$

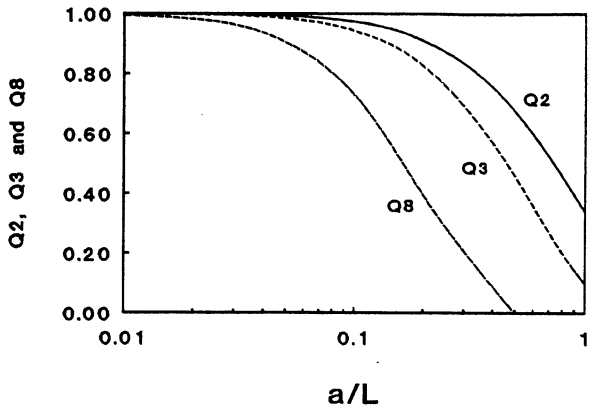


Figure 9: The ratios Q_m of the deflections W(r) and W_o in the Westergaard Model as a function of a/L.

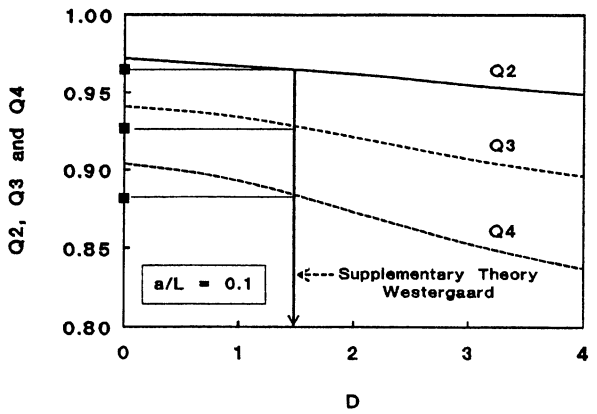


Figure 10: The ratios Q_m of the deflections W(r) and W_o in the Pasternak Model as a function of D for a/L = 0.1.

- Pasternak foundation: In this case the deflections will depend on three parameters (k, a/L and D). The ratio Q_n of the central deflection W_o for a plate diameter a_1 and the central deflection W_o for a plate diameter $a_n = n \times a_1$ is given by equation 18 if the same contact stress σ_o is applied.

$$Q_n = \frac{\int_0^\infty \frac{J_1(x\frac{a_1}{L})}{1 + D^2 x^2 + x^4}\,dx}{n\int_0^\infty \frac{J_1(x\,n\frac{a_1}{L})}{1 + D^2 x^2 + x^4}\,dx} \qquad (18)$$

This ratio depends on the parameters n, a_1/L and D. As shown in figure 2 the ratio of the deflection W_o in the Pasternak model and the deflection W_o in the Westergaard model hardly depends on the parameter a/L. Therefore it is allowed to use equation 16 instead of equation 18. In figure 8 the deviation is shown for a value D = 4. Nevertheless, the obtained curve implies that for a/L values smaller than 0.1 this approach will not lead to a value for the parameter a/L.
Also the ratio Q_m of the deflection W(r) ($r = m \times a_1$) and the central deflection W_o hardly depends on the parameter D as shown in the figures 10 and 11 for two a/L values.

This implies that for the determination of the parameters k, L and D a curve fitting method must be carried out on the whole measured deflection bowl.

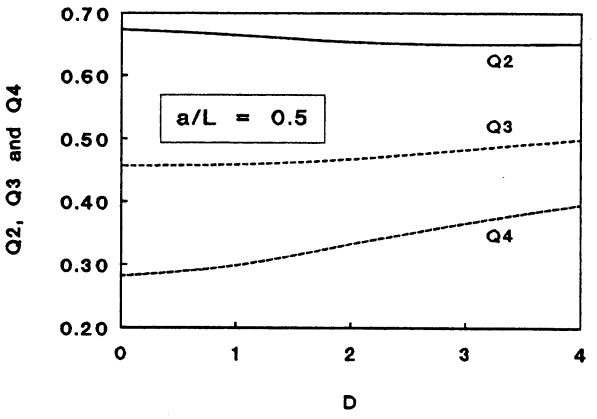


Figure 11: The ratios Q_m of the deflections W(r) and W_o in the Pasternak Model as a function of D for a/L = 1.0.

Finally in figure 10 also the deflection ratios Q_2, Q_3 and Q_4 (W(r)/W(0)) are given for the extreme case of the Supplementary Theory of Westergaard in which the foundation 'reacts' as a Burmister layer (Jeuffroy model). It appears that these ratios are obtained in the Pasternak model for a D value of ca. 1.5 in accordance with the central deflection (figure 2). Therefore it can be concluded that with a D value of 1.5 the Pasternak Model closely resembles the deflectionbowl in the Supplementary Theory of Westergaard for the extreme case.

6. Conclusions

1. The Pasternak foundation is an atractive alternative for the Winkler foundation.
2. With the Pasternak foundation a whole range of possible subgrade reactions can be simulated including the reaction of a linear elastic halfspace (with regard to deflection).

Acknowledgement

Special thanks to Prof. F. van Cauwelaert for the development of computer programs for the calculation of the integrals.

References

1. Westergaard,H.M.,"Stresses in Concrete Pavements Computed by Theoretical Analysis",Public Roads (7), 1926, no.2.
2. Westergaard,H.M.,"Analytical Tools for Judging Results of Structural Tests of Concrete Pavements", Public Roads (14), 1933,no.10.
3. Jeuffroy,G.,"Note sur le Comportement des Chausse´es", Annales des Ponts et Chausse´es, 1959,no.3.
4. Pronk,A.C.,"AFurther Investigation into the Westergaard Formulas for Interior Loading and the (Mis)use of the Modulus of the Subgrade Reaction (k)", 2nd Int.Workshop on Theoretical Design of Concrete Pavements, 1990, Siguenza, Spain
5. Shell Pavement Design Manual,Shell Int.Petroleum Comp. Lmt.,London,1978
6. Zukang Yao,"Design Theory and Procedures of Concrete Pavements in China",2nd Int.Workshop on Theoretical Design of Concrete Pavements, 1990, Siguenza, Spain
7. Holl,D.,"Thin Plates on Elastic Foundation",Proc.of the 5th Int.Congres for Applied Mechanics, Cambridge, Maas,1938,John Wiley&Sons Inc.,New York, N.Y.,1939,pp.71-74.
8. Cauwelaert,F.van,"Bessel and Other Intgral Transforms in Civil Engineering",Report 7-92-411-2, Delft Univ.ofTechn., 1992
9. Watson,G.N.,"A Treatise on the Theory of Bessel Functions",Cambridge Univ. Press, 2nd ed.,Reprint 1980, pp.430,eq.(6).
10. Ioannides,A.M.,"Dimensional Analysis in NDT Rigid Pavement Evaluation", Journ.of Transp.Engin.,ASCE., Vol.(116),no.1, 1990 pp.23-36.
11. Pronk,A.C.,"Solutions for the infinite and finite plate resting on a Winkler foundation and using the Hankel-Bessel transformation",DWW note TW-N-90-90, Road and Hydraulic Eng.Div.,Delft,1990

APPENDIX I Hankel-Bessel Transformation

The Hankel-Bessel Transformation is very useful in case of axial-symmetriccally loaded plates on elastic foundations [7]. A short overview will be given here. The general differential equation for a (thin) plate resting on an elastic foundation is given by equation 19:

$$\left\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right\}\times\left\{\frac{d^2}{dr^2}W(r)+\frac{1}{r}\frac{d}{dr}W(r)\right\} = \frac{-q_v(r)+\sigma(r)}{N} \quad (19)$$

in which $\sigma(r)$ is the axial symmetrical load distribution over the plate and $q_v(r)$ the reaction of the foundation (in case of a Winkler foundation $q_v(r) = k.W(r)$).
The general solution of equation 19 is given by Holl [7]:

$$W(r) = \int_0^\infty \frac{x\,P_o(x)\,K(x)\,J_o(xr)}{1+N\,x^4\,K(x)}\,dx \quad (20)$$

This general solution gives the opportunity to obtain different loadings and different type of subgrade reactions:

$$P_o(x) = \frac{P}{2\pi} \quad (21)$$

for a concentrated load P

$$P_o(x) = \frac{P}{\pi\,a\,x}\,J_1(xa) \quad (22)$$

for an uniformly distributed load P on a circle with radius a

$$P_o(x) = \frac{2P}{\pi\,a^2\,x^2}\,J_2(x\,a) \quad (23)$$

for a parabolic distributed load P on a circle with radius a

$$P_o(x) = \frac{P}{x\sqrt{2\pi\,a\,x}}\,J_{3/2}(x\,a)$$

for a hemispherical distributed load P on a circle with radius a

(24)

$$P_o(x) = p_o\,a\,J_o(x\,a)$$

for a line load of constant intensity p_o on a circumference of a circle

(25)

$$K(x) = \frac{1}{k}$$

for a Winkler foundation : $q_v(r) = k.W(r)$
Herz/Westergaard model

(26)

$$K(x) = \frac{2\,(1-\nu_o^2)}{x\,E_o}$$

for an elastic isotropic subgrade (E_o, ν_o)
no friction between plate and subgrade
(Hogg/Jeuffroy model; no shearing force)

(27)

$$K(x) = \frac{(1+\nu_o)\ (3-4\nu_o)}{2x\ (1-\nu_o)\ E_o}$$

for an elastic isotropic subgrade (E_o, ν_o)
friction between plate and subgrade
(Hogg model ; no radial displacement)

(28)

Using the Hankel-Bessel Transformation it is quite easy to obtain the solution for the finite plate with radius R resting on a Winkler foundation and loaded by a homogeneous distributed load over a circle with radius a [11]. This solution is given equation 29:

$$W(r) = \frac{P}{\pi\, a\, k\, L} \times$$

$$\int_0^\infty \frac{J_1(x\frac{a}{L}) + \cos(x\,\frac{R}{L})}{1 + x^4}\ J_0(x\frac{r}{L})\ dx \qquad \textbf{(29)}$$

Note: $\cos(x\frac{R}{L}) = \sqrt{\frac{\pi R x}{2L}}\ J_{-1/2}(x\frac{R}{L})$

The Jeuffroy model in closed integral form is given by:

$$W(r) = \frac{P\ 2\,(1-\nu_o^2)}{\pi\ a\ E_o} \times$$

$$\int_0^\infty \frac{J_1(x\frac{a}{L_o})\ J_o(x\frac{r}{L_o})}{x\ (1 + x^3)}\ dx \qquad \textbf{(30)}$$

$$\sigma_r(o) = \frac{3\,(1+\nu)\ P\ L_o}{\pi\ a\ h^2} \times$$

$$\int_0^\infty \frac{x\ J_1(x\frac{a}{L_o})}{1 + x^3}\ dx \qquad \textbf{(31)}$$

with $L_o^3 = \frac{2\ (1-\nu_o^2)}{E_o\ N}$